EP 1 '44 B

Mar. '47
V. 18 # 6

THE QUARTERLY JOURNAL OF MATHEMATICS

OXFORD SERIES

Volume 15 *Nos.* 57–8 *Mar.–June* 1944

CONTENTS

Fu-Traing Wang: Some Remarks on Oscillating Series 1

H. C. Lee: On Plane Factorizations of Pseudo-Euclidean Rotations 7

A. P. Guinand: Functional Equations and Self-Reciprocal Functions connected with Lambert Series 11

B. Segre: A Remark on Unicursal Curves Lying on the General Quartic Surface . . . 24

J. G. Semple: Properties of Certain Cubic Primals . 26

W. W. Sawyer: A Property of Certain Differential Equations 34

E. C. Titchmarsh: An Extension of the Sturm-Liouville Expansion 40

OXFORD
AT THE CLARENDON PRESS
1944

Price 15*s. net*

PRINTED IN GREAT BRITAIN BY JOHN JOHNSON AT THE OXFORD UNIVERSITY PRESS

THE QUARTERLY JOURNAL OF MATHEMATICS

OXFORD SERIES

Edited by T. W. CHAUNDY, U. S. HASLAM-JONES, J. H. C. THOMPSON

With the co-operation of A. L. DIXON, W. L. FERRAR, G. H. HARDY, E. A. MILNE, E. C. TITCHMARSH

THE QUARTERLY JOURNAL OF MATHEMATICS (OXFORD SERIES) is published at 7s. 6d. net for a single number with an annual subscription (for four numbers) of 27s. 6d. post free.

While war-time restrictions on production continue subscribers are asked to accept two double numbers per annum.

Papers, of a length normally not exceeding 20 printed pages of the Journal, are invited on subjects of Pure and Applied Mathematics, and should be addressed 'The Editors, Quarterly Journal of Mathematics, Clarendon Press, Oxford'. While every care is taken of manuscripts submitted for publication, the Publisher and the Editors cannot hold themselves responsible for any loss or damage. Authors are advised during the emergency to retain a copy of anything they may send for publication. Authors of papers printed in the Quarterly Journal will be entitled to 50 free offprints. Correspondence on the *subject-matter* of the Quarterly Journal should be addressed, as above, to 'The Editors', at the Clarendon Press. All other correspondence should be addressed to the Publisher (Humphrey Milford, Oxford University Press, at the temporary address).

HUMPHREY MILFORD
OXFORD UNIVERSITY PRESS
AMEN HOUSE, LONDON, E.C.4

Temporary address:
SOUTHFIELD HOUSE, HILL TOP ROAD, OXFORD

THE QUARTERLY JOURNAL
OF
MATHEMATICS
OXFORD SERIES

VOLUME XV

OXFORD
AT THE CLARENDON PRESS
1944

OXFORD UNIVERSITY PRESS
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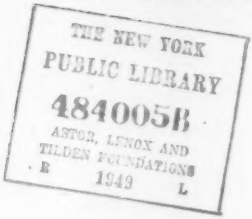

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SOME REMARKS ON OSCILLATING SERIES

By FU-TRAING WANG (*Chekiang*)

[Received 5 September 1942]

[Owing to the difficulty of communication with China this paper has been substantially revised and put into its present form by the kindness of Dr. L. S. Bosanquet without consultation with the author. Parts of the original manuscript have been omitted, and § 3 has been added. The proof of Lemma 5 was obtained by Dr. C. M. Rigby.]

1. Introduction

RECENTLY Boas* proved that *if* $\sigma_n^1 - s = o(n^{\delta-1})$ $(0 < \delta < 1)$ *and* $a_n > -Kn^{-\delta}$, *then the series* $\sum a_n$ *is convergent to sum* s. Here

$$\sigma_n^r = \left\{1 \Big/ \binom{n+r}{n}\right\} \sum_{\nu=0}^{n} \binom{n-\nu+r}{n-\nu} a_\nu$$

is the nth Cesàro mean of order r of the series $a_0+a_1+\ldots$.

By a similar method we can prove the following theorem.†

THEOREM 1. *If* r *is a positive integer,* $0 < \delta < 1$, *and*

$$\text{(i)} \quad \sigma_n^r - s = o\{n^{r(\delta-1)}\} \quad (n \to \infty),$$

$$\text{(ii)} \quad a_n > -Kn^{-\delta},$$

then $\sum a_n$ *is convergent to sum* s.

In this note we show by means of examples taken from the theory of Fourier series that Theorem 1 is a 'best possible' result. More precisely we prove the following theorem.

THEOREM 2. *If* $0 < \delta < \delta' < 1/r$, *then there exists a divergent series* $\sum a_n$ *such that*

$$\text{(i)} \quad a_n = O(n^{-\delta}),$$

$$\text{(ii)} \quad \sigma_n^r = o\{n^{r(\delta'-1)}\}$$

as $n \to \infty$.

* R. B. Boas, Jr., *Duke Math J.*, 4 (1938), 227–30.

† A method of H. D. Kloosterman, *J. of London Math. Soc.*, 15 (1940), 91–6, gives the following more general result:

If (i) $\sigma_n^r - s = o(n^\beta)$ $(\beta > -r)$, (ii) $a_n > -Kn^{-\delta}$ $(\delta \leqslant 1)$

then $s_n - s = o\{n^{(\beta+r(1-\delta))/(r+1)}\}$.

We shall require the following result proved elsewhere.*

LEMMA 1. *If* $0 < \delta < \delta' < 1$, *there is an even integrable function* $\phi(t) \sim \sum a_n \cos nt$ *such that*

(i) $\phi(t) = o(1) \quad (t \to 0)$,

(ii) $\Phi_1(t) = \int_0^t \phi(u)\,du = o(t^{1/\delta'}) \quad (t \to 0)$,

(iii) $a_n = O(n^{-\delta})$,

(iv) $\sum a_n$ *is divergent.*

2. Proof of Theorem 2 (the cases $r = 1$ and 2)

LEMMA 2. *If*

(i) $\phi(t) = o(1)$, (ii) $\Phi_1(t) = o(t^{1/\delta})$

as $t \to 0$, *where* $0 < \delta < 1$, *then* $\sigma_n^1 = o(n^{\delta-1})$.

We have

$$\sigma_n^1 = \int_0^{\pi} \phi(t)\, K_n^1(t)\,dt = \int_0^{n^{-\delta}} + \int_{n^{-\delta}}^{\pi} = I_1 + I_2,$$

where $$K_n^1(t) = \frac{1-\cos(n+1)t}{2\pi(n+1)\sin^2 \frac{1}{2}t} = \begin{cases} O(t^{-1}), \\ O(n^{-1}t^{-2}), \end{cases}$$

and $$\frac{d}{dt}K_n^1(t) = O(t^{-2}).$$

Hence, by (ii),

$$I_1 = \Phi_1(n^{-\delta})K_n^1(n^{-\delta}) - \int_0^{n^{-\delta}} \Phi_1(t)\frac{d}{dt}K_n^1(t)\,dt$$

$$= o(n^{-1})O(n^{\delta}) + \int_0^{n^{-\delta}} o(t^{1/\delta-2})\,dt$$

$$= o(n^{\delta-1}) + o(n^{\delta-1}),$$

while, by (i), $$I_2 = O(n^{-1}) \int_{n^{-\delta}}^{\pi} o(t^{-2})\,dt = o(n^{\delta-1}).$$

The case $r = 1$ of Theorem 2 follows immediately from Lemmas 1 and 2.

* F. T. Wang, *Proc. London Math. Soc.* (2) 47 (1942), 308–25. We may take $\phi(t) = c_n \sin \lambda_n t$ in (α_n, α_{n-1}), where $\lambda_n = n!$, $c_n = \{\log(n+1)\}^{-\frac{1}{2}}$, $\alpha_n = (n!)^{-\delta'}$ for $n = 1, 2, \ldots$, $\alpha_0 = \pi$. By slightly modifying the α_n the function may be made continuous.

LEMMA 3. *If*

$$\text{(i)}\quad \phi(t) = o(1), \qquad \text{(ii)}\quad \Phi_1(t) = o(t^{1/\delta})$$

as $t \to 0$, *where* $0 < \delta < \frac{1}{2}$, *then*

$$\sigma_n^2 = \frac{2n+3}{\pi(n+1)(n+2)} \int_0^\pi \frac{\phi(t)}{2\sin^2 \frac{1}{2}t}\, dt + o\{n^{2(\delta-1)}\},$$

*where the integral on the right-hand side is a Cauchy integral.**

We have
$$\sigma_n^2 = \int_0^\pi \phi(t) K_n^2(t)\, dt,$$

where
$$K_n^2(t) = \frac{2n+3}{2\pi(n+1)(n+2)\sin^2 \frac{1}{2}t} - \frac{\sin(n+\frac{3}{2})t}{2\pi(n+1)(n+2)\sin^3 \frac{1}{2}t}$$

$$= \frac{2n+3}{2\pi(n+1)(n+2)\sin^2 \frac{1}{2}t} + W_n(t),$$

and
$$W_n(t) = \begin{cases} O(n^{-1}t^{-2}), \\ O(n^{-2}t^{-3}), \end{cases}$$

while
$$\frac{d}{dt} W_n(t) = O(n^{-1}t^{-3}).$$

Then

$$\sigma_n^2 - \frac{2n+3}{\pi(n+1)(n+2)} \int_0^\pi \frac{\phi(t)}{2\sin^2 \frac{1}{2}t}\, dt = \left\{ \int_0^{n^{-\delta}} + \int_{n^{-\delta}}^\pi \right\} \phi(t) W_n(t)\, dt = I_1 + I_2,$$

and, by (ii),

$$I_1 = \Phi_1(n^{-\delta}) W_n(n^{-\delta}) - \int_0^{n^{-\delta}} \Phi_1(t) \frac{d}{dt} W_n(t)\, dt$$

$$= o(n^{-1}) O(n^{2\delta-1}) + O(n^{-1}) \int_0^{n^{-\delta}} o(t^{1/\delta-3})\, dt$$

$$= o(n^{2\delta-2}) + o(n^{2\delta-2}),$$

while, by (i)
$$I_2 = O(n^{-2}) \int_{n^{-\delta}}^\pi o(t^{-3})\, dt = o(n^{2\delta-2}).$$

* Plainly

$$\int_\epsilon^\pi \frac{\phi(t)}{2\sin^2 \frac{1}{2}t}\, dt = \left[\frac{\Phi_1(t)}{2\sin^2 \frac{1}{2}t}\right]_\epsilon^\pi - \int_\epsilon^\pi \Phi_1(t) \frac{d}{dt}\left(\frac{1}{2\sin^2 \frac{1}{2}t}\right) dt$$

tends to a limit as $\epsilon \to +0$, since $\Phi_1(t) = o(t^{1/\delta})$ and $1/\delta > 2$.

If we now take the Fourier series defined by Lemma 1, with $0 < \delta < \delta' < \frac{1}{2}$, and replace $\phi(t)$ by $\phi(t) - A(1-\cos t)$, where

$$A = \frac{1}{\pi}\int_0^\pi \frac{\phi(t)}{2\sin^2 \frac{1}{2}t}\,dt,$$

it follows from Lemma 3 that the new Fourier series is such that $\sigma_n^2 = o\{n^{2(\delta'-1)}\}$, while $a_n = O(n^{-\delta})$, which establishes the case $r = 2$ of Theorem 2.

3. Proof of Theorem 2 (the general case)

LEMMA 4. *If*

$$\text{(i)}\quad \phi(t) = o(1), \qquad \text{(ii)}\quad \Phi_1(t) = o(t^{1/\delta})$$

as $t \to 0$, *where* $0 < \delta < 1/r$, *then*

$$\sigma_n^r = \sum_{\nu=1}^{[\frac{1}{2}r]} c_{\nu,n} \int_0^\pi \frac{\phi(t)}{(2\sin \frac{1}{2}t)^{2\nu}}\,dt + o\{n^{r(\delta-1)}\},$$

where $c_{\nu,n} \sim c_\nu n^{1-2\nu}$ $(n \to \infty)$ *and the integrals on the right-hand side are Cauchy integrals.*

We have
$$\sigma_n^2 = \int_0^\pi \phi(t) K_n^r(t),$$

where
$$K_n^r(t) = \sum_{\nu=1}^{[\frac{1}{2}r]} \frac{c_{\nu,n}}{(2\sin \frac{1}{2}t)^{2\nu}} + W_n(t)$$

and

$$\tfrac{1}{2}\pi(-1)^{[\frac{1}{2}r]}\binom{n+r}{n}(2\sin \tfrac{1}{2}t)^{r+1} W_n(t) = \begin{cases} 1-\cos\{n+\frac{1}{2}(r+1)\}t & (r \text{ odd}), \\ \sin\{n+\frac{1}{2}(r+1)t\} & (r \text{ even}), \end{cases}$$

so that
$$W_n(t) = \begin{cases} O(n^{1-r}t^{-r}), \\ O(n^{-r}t^{-r-1}), \end{cases}$$

and
$$\frac{d}{dt}W_n(t) = O(n^{1-r}t^{-r-1}).$$

Then

$$\sigma_n^r - \sum_{\nu=1}^{[\frac{1}{2}r]} \int_0^\pi \frac{\phi(t)}{(2\sin \frac{1}{2}t)^{2\nu}}\,dt = \left\{\int_0^{n^{-\delta}} + \int_{n^{-\delta}}^\pi\right\}\phi(t) W_n(t)\,dt = I_1 + I_2,$$

and, by (ii),

$$I_1 = \Phi_1(n^{-\delta})W_n(n^{-\delta}) - \int_0^{n^{-\delta}} \Phi_1(t)\frac{d}{dt}W_n(t)\,dt$$

$$= o(n^{-1})O(n^{r\delta+1-r}) + O(n^{1-r})\int_0^{n^{-\delta}} o(t^{1/\delta-r-1})\,dt$$

$$= o(n^{r\delta-r}) + o(n^{r\delta-r}),$$

while, by (i),

$$I_2 = O(n^{-r})\int_{n^{-\delta}}^{\pi} o(t^{-r-1})\,dt = o(n^{r\delta-r}).$$

LEMMA 5. *If* $a_0 = 1$ *and*

$$a_n = \frac{1.3....(2n-1)}{2.4....2n}2^n$$

for $n = 1, 2,...,$ *then*

$$D_n \equiv \begin{vmatrix} a_0 & a_1 & . & . & . & a_n \\ a_1 & a_2 & . & . & . & a_{n+1} \\ . & . & . & . & . & . \\ a_n & a_{n+1} & . & . & . & a_{2n} \end{vmatrix} \neq 0$$

for $n = 1, 2,....$

Suppose that $D_m = 0$, where m is a positive integer. Then b_0, $b_1,...,$ b_m can be found, not all zero, such that

$$a_\nu b_0 + a_{\nu+1}b_1 + ... + a_{\nu+m}b_m = 0$$

for $\nu = 0, 1,..., m$.

Now

$$\sum_{r=0}^{\infty} a_r x^r = (1-2x)^{-\frac{1}{2}},$$

and hence, for $|x| < \frac{1}{2}$,

$$\Big(\sum_{r=0}^{\infty} a_r x^r\Big)\Big(\sum_{s=0}^{m} b_s x^{-s}\Big) = \sum_{\nu=1}^{m}(a_0 b_\nu + a_1 b_{\nu+1} + ... + a_{m-\nu}b_m)x^{-\nu} + \\ + \sum_{\nu=m+1}^{\infty}(a_\nu b_0 + a_{\nu+1}b_1 + ... + a_{\nu+m}b_m)x^\nu,$$

from which it follows that

$$\Big(\sum_{s=0}^{m} b_s x^{-s}\Big)^2 = (1-2x)\Big\{\sum_{\nu=1}^{m}(a_0 b_\nu + a_1 b_{\nu+1} + ... + a_{m-\nu}b_m)x^{-\nu} + \\ + \sum_{\nu=m+1}^{\infty}(a_\nu b_0 + a_{\nu+1}b_1 + ... + a_{\nu+m}b_m)x^\nu\Big\}^2.$$

Equating the terms independent of x, and the coefficients of x^{-1}, we have $b_0^2 = 0$ and

$$2b_0b_1 = -2(a_0b_1+a_1b_2+\ldots+a_{m-1}b_m)^2.$$

Hence
$$a_0b_1+a_1b_2+\ldots+a_{m-1}b_m = 0,$$

and it follows that $b_1, b_2,\ldots, b_m$ can be found, not all zero, such that

$$a_\nu b_1+a_{\nu+1}b_2+\ldots+a_{\nu+m-1}b_m = 0$$

for $\nu = 0, 1,\ldots, m-1$. Hence $D_{m-1} = 0$, which leads to a contradiction, since $D_0 \neq 0$.

LEMMA 6. *If $\alpha_0, \alpha_1,\ldots, \alpha_n$ is a given set of numbers, then numbers $\beta_0, \beta_1,\ldots, \beta_n$ can be found such that*

$$\frac{1}{\pi}\int_0^\pi \left\{\sum_{\nu=0}^n \beta_\nu(1-\cos t)^{\nu+\mu}\right\} dt = \alpha_\mu$$

for $\mu = 0, 1,\ldots, n$.

The last equation may be written

$$\sum_{\nu=0}^n a_{\nu+\mu}\beta_\nu = \alpha_\mu$$

for $\mu = 0, 1,\ldots, n$, where $a_0, a_1,\ldots, a_{2n}$ are defined as in Lemma 5, and the result follows from Lemma 5.

If we now take the Fourier series defined by Lemma 1, with $0 < \delta < \delta' < 1/r$, and replace $\phi(t)$ by

$$\phi(t) - \sum_{\nu=0}^{[\frac{1}{2}r]-1} \beta_\nu(1-\cos t)^{[\frac{1}{2}r]+\nu},$$

where the β_ν are chosen, by Lemma 6, so that

$$\frac{1}{\pi}\int_0^\pi \left\{\sum_{\nu=0}^{[\frac{1}{2}r]-1} \beta_\nu(1-\cos t)^{\nu+\mu}\right\} dt = \frac{1}{\pi}\int_0^\pi \frac{\phi(t)}{(1-\cos t)^{[\frac{1}{2}r]-\mu}}\, dt$$

for $\mu = 0, 1,\ldots, [\frac{1}{2}r]-1$, it follows from Lemma 4 that the modified Fourier series is such that $\sigma_n^r = o\{n^{r(\delta'-1)}\}$, while $a_n = O(n^{-\delta})$, which establishes the general case of Theorem 2.

ON PLANE FACTORIZATIONS OF PSEUDO-EUCLIDEAN ROTATIONS

By H. C. LEE [LI HUA-TSUNG] (*Chiating*)

[Received 22 July 1943]

IT is well known that any rotation in the ordinary Euclidean space of three dimensions can be brought about by a succession of three plane rotations. This classical theorem of Euler has been generalized to the ordinary Euclidean space R_n of n dimensions by R. Brauer,* who showed that every rotation in R_n can be factorized into a product of plane rotations. In this paper I shall show that an analogous theorem also holds when R_n is pseudo-Euclidean, i.e. when the fundamental form of R_n is indefinite. For this latter case Brauer's proof cannot be carried through without modifications.

It suffices to confine ourselves to an orthogonal coordinate system of R_n, for which the fundamental form is†

$$x_1^2+\ldots+x_p^2-x_{p+1}^2-\ldots-x_n^2, \tag{1}$$

where p is the index of the form. A homogeneous linear transformation on the variables $x_1,\ldots, x_n$ is called an *orthogonal transformation* of R_n if it leaves the form (1) invariant, i.e. if its matrix A is such that

$$A'GA = G \tag{2}$$

where A' denotes the transpose of A, and G represents the n-rowed diagonal matrix whose first p diagonal elements are $+1$ and the remaining diagonal elements are -1. It follows from (2) that A has determinant ± 1 and that the totality of orthogonal transformations of R_n forms a group. An orthogonal transformation of R_n of determinant $+1$ is called a *rotation* of R_n. It is clear that the totality of rotations of R_n again forms a group.

A rotation of R_n is called a *plane rotation* if it leaves unaffected $n-2$ members of the n variables $x_1,\ldots, x_n$. Every plane rotation which affects the two variables x_i, x_j $(i<j)$ is either of the form

$$\left.\begin{aligned} y_i &= x_i\cos\theta - x_j\sin\theta \\ y_j &= x_i\sin\theta + x_j\cos\theta \\ y_k &= x_k \quad (k\neq i,j) \end{aligned}\right\} \tag{3}$$

* See F. D. Murnaghan, *The Theory of Group Representations* (1938), 320.

† All variables and constants are supposed to be real.

(when i, j are both in the range $1,\ldots, p$ or both in the range $p+1,\ldots, n$), or of the forms

$$\left.\begin{aligned} y_i &= x_i \cosh\theta + x_j \sinh\theta \\ y_j &= x_i \sinh\theta + x_j \cosh\theta \\ y_k &= x_k \quad (k \neq i, j) \end{aligned}\right\}, \tag{4}$$

$$\left.\begin{aligned} y_i &= -x_i \cosh\theta - x_j \sinh\theta \\ y_j &= -x_i \sinh\theta - x_j \cosh\theta \\ y_k &= x_k \quad (k \neq i, j) \end{aligned}\right\} \tag{4'}$$

(when i is in the range $1,\ldots, p$ and j in the range $p+1,\ldots, n$). I shall not make use of plane rotations of the form (4′) and I shall denote the matrix of the plane rotation (3) or (4) by $P_{i,j}(\theta)$. Evidently we have

$$P_{i,j}^{-1}(\theta) = P_{i,j}(-\theta). \tag{5}$$

When distinction is made, I refer to (3) as an *ordinary* plane rotation and (4) a *pseudo* plane rotation. My object is to prove the

THEOREM. *Every rotation of an n-dimensional pseudo-Euclidean space of index p can be factorized into a product of $\frac{1}{2}n(n-1)$ plane rotations of which the number of pseudo members is equal to the smaller of p, $n-p$.*

Let A be any rotation of R_n and let its (λ, μ) element (i.e. the element in the λth row and μth column) be denoted by a_μ^λ. The proof consists in the following three kinds of operations.

1. Make the $(1, 2)$ element of $AP_{1,2}(\theta_2)$ vanish by setting

$$-a_1^1 \sin\theta_2 + a_2^1 \cos\theta_2 = 0,$$

an equation always solvable for θ_2 (to modulus π). The $(1, 2)$ element of $AP_{1,2}(\theta_2)P_{1,3}(\theta_3)$ still vanishing, make its $(1, 3)$ element vanish by properly choosing θ_3 (to modulus π). Continuing this process we see that constants $\theta_2,\ldots, \theta_p$ exist (each determined to modulus π) such that the rotation

$$B = AP_{1,2}(\theta_2)\ldots P_{1,p}(\theta_p) \tag{6}$$

has $b_2^1 = \ldots = b_p^1 = 0$ where b_μ^λ denotes the (λ, μ) element of B.* If b_1^1 is negative, we make it positive by replacing θ_p by $\theta_p + \pi$.

2. The rotation $BP_{n-1,n}(\theta_n)$ having its $(1, 1)$ element non-negative and $(1, 2),\ldots, (1, p)$ elements zero, make its $(1, n)$ element vanish by setting $-b_{n-1}^1 \sin\theta_n + b_n^1 \cos\theta_n = 0$, an equation always solvable

* No further process in this direction can be applied since an equation of the form $\alpha \sinh\theta + \beta \cosh\theta = 0$ is not solvable for θ unless $|\alpha| > |\beta|$.

for θ_n (to modulus π). The rotation $BP_{n-1,n}(\theta_n)P_{n-2,n-1}(\theta_{n-1})$ having its $(1,1)$ element non-negative and $(1,2),\ldots,(1,p),(1,n)$ elements zero, make its $(1,n-1)$ element vanish by properly choosing θ_{n-1} (to modulus π). Continuing this new process we see that constants $\theta_n, \theta_{n-1},\ldots,\theta_{p+2}$ exist (each determined to modulus π) such that the rotation

$$C = AP_{1,2}(\theta_2)\ldots P_{1,p}(\theta_p)P_{n-1,n}(\theta_n)\ldots P_{p+1,p+2}(\theta_{p+2}) \tag{7}$$

has all its elements in the first row zero except perhaps c_1^1 (which is non-negative) and c_{p+1}^1, where c_μ^λ denotes the (λ,μ) element of C. As a rotation of R_n, C has its elements in the first row such that

$$(c_1^1)^2+\ldots+(c_p^1)^2-(c_{p+1}^1)^2-\ldots-(c_n^1)^2 = 1,$$

which reduces to
$$(c_1^1)^2-(c_{p+1}^1)^2 = 1 \tag{8}$$

in the case under consideration. Hence $c_1^1 \neq 0$ and, since c_1^1 is non-negative, it is positive. It follows that $c_1^1 > 1$, and consequently a constant θ_{p+1} is determined, except for signs, by $c_1^1 = \cosh\theta_{p+1}$. By (8) we find $c_{p+1}^1 = \pm\sinh\theta_{p+1}$ and we may fix the sign of θ_{p+1} such that

$$c_1^1 = \cosh\theta_{p+1}, \quad c_{p+1}^1 = -\sinh\theta_{p+1}. \tag{9}$$

3. Then the rotation

$$D = CP_{1,p+1}(\theta_{p+1}) \tag{10}$$

has $d_1^1 = 1$, $d_2^1 = \ldots = d_n^1 = 0$, where d_μ^λ denotes the (λ,μ) element of D. Since D, as a rotation of R_n, has its first row (pseudo) orthogonal to every other row, i.e.

$$d_1^1 d_1^\nu+\ldots+d_p^1 d_p^\nu-d_{p+1}^1 d_{p+1}^\nu-\ldots-d_n^1 d_d^\nu = 0 \quad (\nu = 2,\ldots,n),$$

we also have $d_1^2 = \ldots = d_1^n = 0$, so that D is of the form

$$D = \begin{bmatrix} 1 & 0 & . & . & . & 0 \\ 0 & & & & & \\ . & & & & & \\ . & & & D_{n-1} & & \\ . & & & & & \\ 0 & & & & & \end{bmatrix},$$

where D_{n-1} is clearly a rotation of a pseudo-Euclidean space of $n-1$ dimensions.

After these operations we have from (7) and (10)

$$D = AP_{1,2}(\theta_2)\ldots P_{1,p}(\theta_p)P_{n-1,n}(\theta_n)\ldots P_{p+1,p+2}(\theta_{p+2})P_{1,p+1}(\theta_{p+1})$$

which, making use of (5), may be written

$$A = DP_{1,p+1}(\phi_{p+1})P_{p+1,p+2}(\phi_{p+2})\ldots P_{n-1,n}(\phi_n)\times \\ \times P_{1,p}(\phi_p)P_{1,p-1}(\phi_{p-1})\ldots P_{1,2}(\phi_2) \quad (11)$$

where $\phi_\nu = -\theta_\nu$ $(\nu = 2,\ldots,n)$. From the above construction ϕ_p is determined to modulus 2π, ϕ_{p+1} is uniquely determined, and the other ϕ's are determined to modulus π.

On proceeding with D_{n-1} as we did for A (or, equivalently, by induction from $n-1$ to n) we see from (11) that A may be completely factorized into a product of plane rotations. Since the right-hand side of (11) involves $n-1$ P's besides D, the complete factorization of A will involve $(n-1)+(n-2)+\ldots+1 = \frac{1}{2}n(n-1)$ plane rotations. Of the P's on the right of (11) only $P_{1,p+1}$ is a pseudo plane rotation, all others being ordinary plane rotations. It follows that the complete factorization of A involves only p pseudo plane rotations. By rewriting the form (1) with the negative terms first (or by changing the sign of (1)) the number of pseudo plane rotations in the complete factorization of A becomes $n-p$. Thus the theorem is completely proved.

FUNCTIONAL EQUATIONS AND SELF-RECIPROCAL FUNCTIONS CONNECTED WITH LAMBERT SERIES

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[Received 21 August 1943]

1. Introduction

In this paper we discuss some results on the function $\phi(z)$ defined for $R(z) > 0$ by the Lambert series

$$\phi(z) = \sum_{n=1}^{\infty} \frac{1}{e^{2\pi nz} - 1}. \tag{1.1}$$

Alternatively, we have

$$\phi(z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} e^{-2\pi mnz} = \sum_{n=1}^{\infty} d(n) e^{-2\pi nz}, \tag{1.2}$$

where $d(n)$ is the number of divisors of n.

The results were suggested by a formal application of Voronoï's summation formula outlined in the next section. In the last section the results are extended to sums involving $\sigma_k(n)$, the sum of the kth powers of the divisors of n.

2. Formalities

Voronoï's summation formula* may be written in the form

$$-\tfrac{1}{4}f(+0) + \sum_{n=1}^{\infty} d(n) f(n) = -\tfrac{1}{4}g(+0) + \sum_{n=1}^{\infty} d(n) g(n), \tag{2.1}$$

where

$$g(x) = 2\pi \int_0^{\infty} f(t) M_0(4\pi x^{\frac{1}{2}} t^{\frac{1}{2}})\, dt, \tag{2.2}$$

$$M_0(z) = -Y_0(z) + \frac{2}{\pi} K_0(z),$$

and $f(x)$ satisfies appropriate conditions.

Now† $$\cos 2\pi x = 2\pi \int_0^{\infty} \cos 2\pi t\, M_0(4\pi x^{\frac{1}{2}} t^{\frac{1}{2}})\, dt.$$

* G. Voronoï, *Ann. École norm. sup.* (3), 21 (1904), 207–67, 459–533, or A. L. Dixon and W. L. Ferrar, *Quart. J. of Math.* (Oxford), 8 (1937), 66–74. The formula is usually given in an unsymmetrical form, but (2.1) is easily deduced if assumptions are made about $g(x)$.

† J. R. Wilton, *J. für Math.* 169 (1932), 219–37 (224).

That is, $\cos 2\pi x$ is self-reciprocal with respect to the transformation (2.2), and consequently, if we put

$$f(x) = \cos 2\pi xy,$$

then

$$g(x) = \frac{1}{y}\cos\frac{2\pi x}{y}.$$

Substituting these functions in (2.1) we get the formal result

$$-\tfrac{1}{4}+\sum_{n=1}^{\infty}{}' d(n)\cos 2\pi ny = -\frac{1}{4y}+\frac{1}{y}\sum_{n=1}^{\infty} d(n)\cos\frac{2\pi n}{y}. \qquad (2.3)$$

Obviously neither series converges, but this suggests that (2.3) may be interpreted in some exact sense. Wilton* has proved an approximate functional equation for finite sums

$$\sum_{n<\omega} d(n)\cos 2\pi ny,$$

which corresponds to (2.3), and he has proved that for rational y this partial sum behaves like

$$A\omega\log\omega+O(\omega)$$

as $\omega\to\infty$, where A depends only on y. Hence we cannot in general associate finite values with the separate sums

$$\sum_{n=1}^{\infty}{}' d(n)\cos 2\pi ny, \qquad \frac{1}{y}\sum_{n=1}^{\infty} d(n)\cos\frac{2\pi n}{y}$$

by any ordinary methods of summation.

Now (2.3) may also be written in the form

$$-\tfrac{1}{2}+\sum_{n=1}^{\infty} d(n)(e^{2\pi iny}+e^{-2\pi iny}) = -\frac{1}{2y}+\frac{1}{y}\sum_{n=1}^{\infty} d(n)(e^{2\pi in/y}+e^{-2\pi in/y}),$$

or, using (1.2) formally,

$$-\tfrac{1}{2}+\phi(iy)+\phi(-iy) = -\frac{1}{2y}+\frac{1}{y}\phi\left(\frac{i}{y}\right)+\frac{1}{y}\phi\left(-\frac{i}{y}\right). \qquad (2.4)$$

In the present paper we show that the function $F(z)$, defined by

$$F(z) = \phi(z)-\frac{1}{2\pi z}(\gamma-\log 2\pi z), \qquad (2.5)$$

* J. R. Wilton, loc. cit. 222. See also S. Wigert, *Acta Math.* 41 (1918), 197–218, and E. Landau, *Archiv der Math. und Phys.* 27 (1918), 144–6.

is self-reciprocal with respect to the Fourier kernel*

$$\frac{2x}{\pi(x^2-1)}.$$

That is, for real positive z,

$$F(z) = \frac{2}{\pi} P \int_0^\infty \frac{zt}{z^2t^2-1} F(t)\, dt, \tag{2.6}$$

where the P indicates that the principal value of the integral at $t = 1/z$ is to be taken.

The corresponding result when z is complex is used to show that the functions

$$F(z)+\frac{i}{z}F\left(\frac{1}{z}\right), \qquad F(z)-\frac{i}{z}F\left(\frac{1}{z}\right)$$

can be continued analytically for $R(z) < 0$. Further, their analytic continuations satisfy a functional equation corresponding to the formal result (2.4).

3. Asymptotic formula and integral equation for $F(z)$

Now $$\zeta^2(s) = \sum_{n=1}^\infty d(n)n^{-s} \qquad \{R(s) > 1\}.$$

Hence, for $c > 1$, $R(z) > 0$,

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta^2(s)(2\pi z)^{-s}\, ds &= \sum_{n=1}^\infty \frac{d(n)}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)(2\pi nz)^{-s}\, ds \\ &= \sum_{n=1}^\infty d(n)e^{-2\pi nz} \\ &= \phi(z),\end{aligned} \tag{3.1}$$

since the series and integral are absolutely convergent.

If $s = \sigma+it$ and $|t| \to \infty$,

$$|\Gamma(s)| \sim (2\pi)^{\frac{1}{2}}|t|^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}\pi|t|},$$

and† $$|\zeta(s)| \begin{cases} \leqslant \zeta(\sigma) & (\sigma > 1), \\ = O(t^{\frac{1}{2}-\frac{1}{2}\sigma}\log|t|) & (0 \leqslant \sigma \leqslant 1), \\ \leqslant t^{\frac{1}{2}-\sigma}\zeta(1-\sigma) & (\sigma < 0). \end{cases} \tag{3.2}$$

* E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, 1937), 219. This kernel also leads to Hilbert transforms (cf. Chapter V).

† E. C. Titchmarsh, *The Zeta-function of Riemann* (Cambridge, 1930), Theorems 1 and 11.

Hence the integral (3.1) converges absolutely for $R(z) > 0$ and any value of c, and we can move the line of integration to the left. The integrand has poles at $s = 1,\ 0,\ 1-2n$. The residue at $s = 1$ is

$$\frac{1}{2\pi z}(\gamma - \log 2\pi z).$$

Hence

$$F(z) = \phi(z) - \frac{1}{2\pi z}(\gamma - \log 2\pi z) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(s)\zeta^2(s)(2\pi z)^{-s}\,ds. \quad (3.3)$$

That is, the Mellin transform of $F(z)$ on $\sigma = \frac{1}{2}$ is

$$\mathfrak{F}(s) = (2\pi)^{-s}\Gamma(s)\zeta^2(s), \quad (3.4)$$

and by (3.2) this belongs to $L^2(-\infty, \infty)$ on $\sigma = \frac{1}{2}$. Hence $F(z)$, $\mathfrak{F}(s)$ are Mellin transforms* of L^2. For complex values of z in $R(z) > 0$ we must take the principal value of $\log z$ in (3.3).

Moving the line of integration further to the left, the residue at $s = 0$ is $\zeta^2(0) = \frac{1}{4}$, and the residues at $s = 1-2n$ are

$$-\frac{\zeta^2(1-2n)}{(2n-1)!}(2\pi z)^{2n-1} = -\frac{B_n^2}{2n(2n)!}(2\pi z)^{2n-1}.$$

Hence we deduce:†

THEOREM 1. *If* $R(z) > 0$, *then*

$$F(z) = \frac{1}{4} - \sum_{n=1}^{N} \frac{B_n^2}{2n(2n)!}(2\pi z)^{2n-1} + \frac{1}{2\pi i}\int_{-2N-i\infty}^{-2N+i\infty} \Gamma(s)\zeta^2(s)(2\pi z)^{-s}\,ds$$

$$= \frac{1}{4} - \sum_{n=1}^{N} \frac{B_n^2}{2n(2n)!}(2\pi z)^{2n-1} + O(z^{2N+1}) \quad (3.5)$$

as $z \to 0$.

Thus on the real axis $F(z)$ behaves like $\frac{1}{4} + O(z)$ near the origin, and like $(\log z)/2\pi z$ as $z \to \infty$. Now by (3.4) and the functional equation of the zeta-function

$$\begin{aligned}\mathfrak{F}(s) &= (2\pi)^{-s}\Gamma(s)\zeta^2(s)\\ &= \{(2\pi)^{-s}\Gamma(s)\zeta(s)\}\{2^s\pi^{s-1}\Gamma(1-s)\sin\tfrac{1}{2}s\pi\,\zeta(1-s)\}\\ &= \tfrac{1}{2}\sec\tfrac{1}{2}s\pi\,\zeta(s)\zeta(1-s). \quad (3.6)\end{aligned}$$

* E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, 1937) 94–5.

† O. Schlömilch, *Zeits. für Math. und Phys.* 6 (1861), 407–15, proves this result for real z. See also S. Wigert and E. Landau, loc. cit.

Now, if we put $\mathfrak{K}(s) = \tan \frac{1}{2}s\pi,$

then $\mathfrak{K}(s)\mathfrak{K}(1-s) = 1,$

and $\mathfrak{F}(s) = \mathfrak{K}(s)\mathfrak{F}(1-s),$

and consequently $F(z)$ is self-reciprocal with respect to the transformation* arising from $\mathfrak{K}(s)$. That is, for real positive z

$$\int_0^z F(t)\,dt = \frac{1}{\pi}\int_0^\infty \log|z^2t^2-1|\,F(t)\frac{dt}{t}.$$

However, if we consider complex z, we can deduce the result in the form (2.6). Suppose $R(z) > 0$, $I(z) > 0$. Then

$$\frac{2xz}{\pi(x^2z^2-1)}$$

belongs to $L^2(0,\infty)$ as a function of x, and its Mellin transform is

$$\frac{2}{\pi}\int_0^\infty \frac{xz}{x^2z^2-1}\,x^{s-1}\,dx = -ie^{\frac{1}{2}s\pi i}\sec\tfrac{1}{2}s\pi\, z^{-s},$$

as we can readily show by contour integration.

Hence, by the Parseval theorem for Mellin transforms† and (3.6),

$$\begin{aligned}
\frac{2}{\pi}\int_0^\infty \frac{xz}{x^2z^2-1}F(x)\,dx &= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} -ie^{\frac{1}{2}s\pi i}\sec\tfrac{1}{2}s\pi\, z^{-s}\mathfrak{F}(1-s)\,ds \\
&= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} (\tan\tfrac{1}{2}s\pi - i)z^{-s}\mathfrak{F}(1-s)\,ds \\
&= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathfrak{F}(s)z^{-s}\,ds - \frac{1}{2\pi i}\frac{i}{z}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathfrak{F}(1-s)z^{1-s}\,ds \\
&= F(z) - \frac{1}{2\pi i}\frac{i}{z}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathfrak{F}(s)z^s\,ds \\
&= F(z) - \frac{i}{z}F\left(\frac{1}{z}\right)
\end{aligned}$$

by (3.3) and (3.4).

* E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, 1937), Chapter VIII.

† E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, 1937), Theorem 72.

Similarly for $R(z) > 0$, $I(z) < 0$ we can show that

$$F(z)+\frac{i}{z}F\left(\frac{1}{z}\right) = \frac{2}{\pi}\int_0^\infty \frac{xz}{x^2z^2-1}F(x)\,dx.$$

In each case the integrand has poles at $\pm 1/z$ with residues

$$\mp\frac{2i}{z}F\left(\frac{1}{z}\right).$$

Hence, by deforming the path of integration, we have

THEOREM 2. *If $F(z)$ is defined for $R(z) > 0$ by (1.1) and (2.5), then*

$$\text{(i)} \qquad F(z)-\frac{i}{z}F\left(\frac{1}{z}\right) = \frac{2}{\pi}\int_C \frac{zt}{z^2t^2-1}F(t)\,dt, \tag{3.7}$$

where C is a contour from 0 to $+\infty$ in the half-plane $R(t) > 0$ which passes above $t = 1/z$;

$$\text{(ii)} \qquad F(z)+\frac{i}{z}F\left(\frac{1}{z}\right) = \frac{2}{\pi}\int_D \frac{zt}{z^2t^2-1}F(t)\,dt, \tag{3.8}$$

where D is a contour from 0 to $+\infty$ in the half-plane $R(t) > 0$ which passes below $t = 1/z$.

In the particular case $I(z) = 0$ we can take the contour C as the straight line from $t = 0$ to $t = (1-\delta)/z$, the semicircle $t = (1+\delta e^{i\theta})/z$ with θ varying from π to 0, and the straight line from $t = (1+\delta)/z$ to $+\infty$. The part arising from the semicircle is

$$\frac{2i\delta}{\pi z}\int_\pi^0 \frac{(1+\delta e^{i\theta})}{(1+\delta e^{i\theta})^2-1}F\{(1+\delta e^{i\theta})/z\}e^{i\theta}\,d\theta$$

$$= -\frac{2i}{\pi z}\int_0^\pi \frac{(1+\delta e^{i\theta})}{(2+\delta e^{i\theta})}F\{(1+\delta e^{i\theta})/z\}\,d\theta.$$

As $\delta \to +0$ this tends to $\quad -\dfrac{i}{z}F\left(\dfrac{1}{z}\right)$.

Hence

$$F(z)-\frac{i}{z}F\left(\frac{1}{z}\right) = -\frac{i}{z}F\left(\frac{1}{z}\right)+\frac{2}{\pi}\lim_{\delta\to+0}\left\{\int_0^{(1-\delta)/z} + \int_{(1+\delta)/z}^\infty \frac{zt}{z^2t^2-1}F(t)\,dt\right\}.$$

That is:

THEOREM 3.* *If z is real and positive, and P indicates that the principal value of the integral is to be taken at $t = 1/z$, then*

$$F(z) = \frac{2}{\pi} P \int_0^\infty \frac{zt}{z^2t^2-1} F(t)\, dt. \tag{3.9}$$

4. Continuation for $R(z) < 0$

Suppose $I(z) > 0$ in (3.7). Then we can put

$$F(z) - \frac{i}{z} F\left(\frac{1}{z}\right) = \frac{2}{\pi} \int_0^\infty \frac{zt}{z^2t^2-1} F(t)\, dt,$$

and the integral converges absolutely for all $I(z) > 0$, including $R(z) \leqslant 0$. Hence it can be differentiated with respect to z, and defines an analytic function in this region; the function is the analytic continuation of

$$\left[F(z) - \frac{i}{z} F\left(\frac{1}{z}\right)\right]. \tag{4.1}$$

We will use square brackets to indicate the analytic continuation of the whole expression within the brackets, as distinguished from the individual terms therein.

If $I(z)$ now passes from positive to negative values in $R(z) < 0$, then the pole at $t = -1/z$ crosses the positive real axis. We can continue the expression (4.1) into the region $R(z) < 0$, $I(z) < 0$ by deforming the contour again, and putting

$$\left[F(z) - \frac{i}{z} F\left(\frac{1}{z}\right)\right] = \frac{2}{\pi} \int_{C_1} \frac{zt}{z^2t^2-1} F(t)\, dt.$$

where C_1 is a contour in $R(t) > 0$ from 0 to $+\infty$ passing above the point $t = -1/z$. However, we cannot use this method to continue (4.1) across the line $R(z) = 0$, $I(z) < 0$, since the integral is not defined for $R(t) < 0$. Furthermore, if we pass the contour below the

* This result is equivalent to an integral equation given by Th. Stieltjes, *Ann. École norm. sup.* (3), 3 (1886), 201–58. In our notation the integral equation is

$$\phi(z) = \frac{1}{2\pi z}(\gamma - \log 2\pi z) + \frac{1}{4} - \frac{2}{\pi z} P \int_0^\infty \frac{t}{1-t^2} \phi(t/z)\, dt.$$

pole at $t = -1/z$, then the residue $\frac{2i}{z} F\left(-\frac{1}{z}\right)$ at this pole must be added, and we find

$$\left[F(z) - \frac{i}{z} F\left(\frac{1}{z}\right) + \frac{2i}{z} F\left(-\frac{1}{z}\right)\right] = \frac{2}{\pi} \int_C \frac{zt}{z^2t^2 - 1} F(t)\, dt.$$

Thus we get the analytic continuation of this function, not of (4.1) itself.

Similar results hold for

$$\left[F(z) + \frac{i}{z} F\left(\frac{1}{z}\right)\right],$$

and we have:

THEOREM 4. *The functions*

$$\left[F(z) - \frac{i}{z} F\left(\frac{1}{z}\right)\right], \qquad \left[\phi(z) - \frac{i}{z} \phi\left(\frac{1}{z}\right)\right]$$

defined by (1.1) *and* (2.5) *for* $R(z) > 0$, *can be continued analytically over the whole z-plane with the exception of a cut from* 0 *to* $-i\infty$.

Similarly, the functions

$$\left[F(z) + \frac{i}{z} F\left(\frac{1}{z}\right)\right], \qquad \left[\phi(z) + \frac{i}{z} \phi\left(\frac{1}{z}\right)\right]$$

can be continued analytically over the whole z-plane with the exception of a cut from 0 *to* $+i\infty$.

5. The functional equation

Putting $-z$ for the argument in (3.8)

$$\left[F(-z) - \frac{i}{z} F\left(-\frac{1}{z}\right)\right] = -\frac{2}{\pi} \int_D \frac{zt}{z^2t^2 - 1} F(t)\, dt, \tag{5.1}$$

where D is a contour from 0 to $+\infty$ in $R(t) > 0$ passing below $t = -1/z$.

Now, if $I(z) > 0$, then $I(-1/z) > 0$, and a contour passing below $t = -1/z$ can also pass above $t = 1/z$, and we can make the contour D identical with the contour C of (3.7). Hence, adding (3.7) and (5.1), we have

$$\left[F(z) - \frac{i}{z} F\left(\frac{1}{z}\right)\right] + \left[F(-z) - \frac{i}{z} F\left(-\frac{1}{z}\right)\right] = 0$$

for $I(z) > 0$. By analytic continuation we have

THEOREM 5. *If the square brackets indicate the analytic continuations of the whole expressions within them, then*

$$\left[F(z)-\frac{i}{z}F\left(\frac{1}{z}\right)\right]+\left[F(-z)-\frac{i}{z}F\left(-\frac{1}{z}\right)\right]=0$$

for all values of z except those on a cut from 0 to $-i\infty$.

In the particular case when $z=iy$, where y does not lie on the negative real axis, we have

$$\left[F(z)-\frac{i}{z}F\left(\frac{1}{z}\right)\right]=\left[\phi(iy)-\frac{1}{y}\phi\left(-\frac{i}{y}\right)\right]-$$

$$-\frac{1}{2\pi iy}\{\gamma-\log 2\pi iy\}-\frac{y}{2\pi i}\{\gamma-\log(-2\pi i/y)\}.$$

Also

$$\left[F(-z)-\frac{i}{z}F\left(-\frac{1}{z}\right)\right]=\left[\phi(-iy)-\frac{1}{y}\phi\left(\frac{i}{y}\right)\right]+$$

$$+\frac{1}{2\pi iy}\{\gamma-\log(-2\pi iy)\}+\frac{y}{2\pi i}\{\gamma-\log(2\pi i/y)\}.$$

Adding and using Theorem 5 we get

$$\left[\phi(iy)-\frac{1}{y}\phi\left(-\frac{i}{y}\right)\right]+\left[\phi(-iy)-\frac{1}{y}\phi\left(\frac{i}{y}\right)\right]+$$

$$+\frac{1}{2\pi iy}\{\log i-\log(-i)\}-\frac{1}{2\pi i}\{\log i-\log(-i)\}=0.$$

The principal values of the logarithms are to be taken, i.e. $\log i=\frac{1}{2}\pi i$, $\log(-i)=-\frac{1}{2}\pi i$, and we have

THEOREM 6. *If $\phi(z)$ is defined by* (1.1), *y is not on the negative real axis, and the convention of Theorem 5 for square brackets is used, then*

$$\left[\phi(iy)-\frac{1}{y}\phi\left(-\frac{i}{y}\right)\right]+\left[\phi(-iy)-\frac{1}{y}\phi\left(\frac{i}{y}\right)\right]=\frac{1}{2}-\frac{1}{2y}.$$

This corresponds to the formal result (2.4).

6. Real form of the functional equation

If we put $z=\pm x+iy$ in Theorem 5 and make $x\to+0$, we can deduce a form of the functional equation in terms of functions of a real variable only.

In this section we give a more direct proof of an equivalent result. If y is real and positive, $0 < \alpha < \frac{1}{2}\pi$, and $\beta = \frac{1}{2}\pi - \alpha$, then

$$e^{-2\pi xy \sin\alpha} \begin{matrix}\cos\\ \sin\end{matrix}(2\pi xy\cos\alpha), \qquad (2\pi y)^{-s}\Gamma(s) \begin{matrix}\cos\\ \sin\end{matrix}\beta s$$

are Mellin transforms of each other for $R(s) > 0$. Hence, if $c > 1$,

$$\sum_{n=1}^{\infty} d(n) e^{-2\pi ny \sin\alpha} \cos(2\pi ny\cos\alpha)$$

$$= \sum_{n=1}^{\infty} \frac{d(n)}{2\pi i} \int_{c-i\infty}^{c+i\infty} (2\pi ny)^{-s}\Gamma(s)\cos\beta s \, ds$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (2\pi y)^{-s}\Gamma(s)\zeta^2(s)\cos\beta s \, ds$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tfrac{1}{2}\cos\beta s \sec\tfrac{1}{2}s\pi \, \zeta(s)\zeta(1-s)y^{-s} \, ds$$

$$= \frac{\cos\beta}{2\pi y}(\gamma - \log 2\pi y) - \frac{\beta\sin\beta}{2\pi y} + \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \tfrac{1}{2}\cos\beta s \sec\tfrac{1}{2}s\pi \, \zeta(s)\zeta(1-s)y^{-s} \, ds, \tag{6.1}$$

on moving the line of integration past the pole at $s = 1$. Similarly,

$$\frac{1}{y}\sum_{n=1}^{\infty} d(n) e^{-2\pi n \sin\alpha/y} \cos(2\pi n\cos\alpha/y + \alpha)$$

$$= \frac{1}{y}\sum_{n=1}^{\infty} d(n) e^{-2\pi n \sin\alpha/y}\{\cos\alpha\cos(2\pi n\cos\alpha/y) - \sin\alpha\sin(2\pi n\cos\alpha/y)\}$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tfrac{1}{2}\sin\beta(1-s)\sec\tfrac{1}{2}s\pi \, \zeta(s)\zeta(1-s)y^{s-1} \, ds$$

$$= \frac{1}{2\pi i} \int_{1-c-i\infty}^{1-c+i\infty} \tfrac{1}{2}\sin\beta s \operatorname{cosec}\tfrac{1}{2}s\pi \, \zeta(s)\zeta(1-s)y^{-s} \, ds$$

$$= -\frac{\beta}{2\pi} + \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \tfrac{1}{2}\sin\beta s \operatorname{cosec}\tfrac{1}{2}s\pi \, \zeta(s)\zeta(1-s)y^{-s} \, ds. \tag{6.2}$$

Subtracting (6.2) from (6.1)

$$\sum_{n=1}^{\infty} d(n)e^{-2\pi ny\sin\alpha}\cos(2\pi ny\cos\alpha) - \\ -\frac{1}{y}\sum_{n=1}^{\infty} d(n)e^{-2\pi n\sin\alpha/y}\cos(2\pi n\cos\alpha/y+\alpha)$$

$$= \frac{\cos\beta}{2\pi y}(\gamma-\log 2\pi y) - \frac{\beta\sin\beta}{2\pi y} + \frac{\beta}{2\pi} + \\ + \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \tfrac{1}{2}\zeta(s)\zeta(1-s)y^{-s}\left\{\frac{\cos\beta s}{\cos\frac{1}{2}s\pi} - \frac{\sin\beta s}{\sin\frac{1}{2}s\pi}\right\} ds$$

$$= \frac{\sin\alpha}{2\pi y}(\gamma-\log 2\pi y) - \cos\alpha\left(\frac{1}{4y}-\frac{\alpha}{2\pi y}\right) + \frac{1}{4} - \frac{\alpha}{2\pi} + \\ + \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\sin\alpha s}{\sin s\pi}\zeta(s)\zeta(1-s)y^{-s}\,ds.$$

Now the integrand is $\quad O\{e^{(\alpha-\frac{1}{2}\pi)|t|}|t|^{\frac{1}{2}}\}.$

Hence the integral converges uniformly as $\alpha \to +0$, and we find

THEOREM 7. *If y is real and positive, then*

$$\lim_{\alpha\to+0}\left\{\sum_{n=1}^{\infty} d(n)e^{-2\pi ny\sin\alpha}\cos(2\pi ny\cos\alpha) - \\ -\frac{1}{y}\sum_{n=1}^{\infty} d(n)e^{-2\pi n\sin\alpha/y}\cos(2\pi n\cos\alpha/y+\alpha)\right\} \\ = \frac{1}{4}-\frac{1}{4y}.$$

7. Extension to sums involving $\sigma_k(n)$

Corresponding results also hold for the function $\phi_k(z)$, defined for real k and $R(z) > 0$ by

$$\phi_k(z) = z^k\sum_{n=1}^{\infty}\frac{n^{2k}}{e^{2\pi nz}-1} = z^k\sum_{n=1}^{\infty}\sigma_{2k}(n)e^{-2\pi nz}. \qquad (7.1)$$

The results may be proved by the methods of the previous sections, so full details are not given.

We have*

$$\zeta(s+k)\zeta(s-k) = \sum_{n=1}^{\infty}\sigma_{2k}(n)n^{-s-k} \qquad \{R(s) > 1+|k|\},$$

* G. Pólya und G. Szegö, *Aufgaben und Lehrsätze* II (Berlin, 1925), 126.

and it follows as in (3.1) that

$$\phi_k(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (2\pi)^{-s-k}\Gamma(s+k)\zeta(s+k)\zeta(s-k)z^{-s}\,ds \quad (c>1+|k|). \tag{7.2}$$

Now the integrand will have a pole on the line $R(s)=\frac{1}{2}$ only if $k=\frac{1}{2}$, $-\frac{1}{2}$, or $-2n+\frac{1}{2}$ ($n=1, 2, 3,\ldots$). All these cases are included in $k=p-\frac{1}{2}$ (p an integer). Assume first that k is not of this form, and let $R_k(z)$ be the sum of the residues of the integrand at poles in the strip $\frac{1}{2}<R(s)\leqslant 1+|k|$. Put

$$F_k(z) = \phi_k(z) - R_k(z). \tag{7.3}$$

Then

$$F_k(z) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathfrak{F}_k(s)z^{-s}\,ds,$$

where

$$\begin{aligned}\mathfrak{F}_k(s) &= (2\pi)^{-s-k}\Gamma(s+k)\zeta(s+k)\zeta(s-k)\\ &= \tfrac{1}{2}\sec\tfrac{1}{2}\pi(s+k)\zeta(1-s-k)\zeta(s-k).\end{aligned}$$

Hence $F_k(z)$ will be self-reciprocal with respect to the Fourier kernel arising from

$$\begin{aligned}\mathfrak{K}_k(s) &= \frac{\mathfrak{F}_k(s)}{\mathfrak{F}_k(1-s)} = \frac{\sin\frac{1}{2}\pi(s-k)}{\cos\frac{1}{2}\pi(s+k)}\\ &= -\sin\pi k+\cos\pi k\tan\tfrac{1}{2}\pi(s+k).\end{aligned}$$

This leads to the transformation

$$g(x) = -\frac{\sin\pi k}{x}f\left(\frac{1}{x}\right)+\frac{2}{\pi}\cos\pi k\,P\int_0^\infty \frac{(xt)^{1+\kappa}}{x^2t^2-1}f(t)\,dt,$$

where $k=2n+\kappa$, n is an integer, and $-\frac{3}{2}<\kappa<\frac{1}{2}$.

Corresponding to Theorems 2, 3, 4, and 5 we find

THEOREM 8. *If $F_k(z)$ is defined for $R(z)>0$ by (7.1) and (7.3), k is real, and $k+\frac{1}{2}$ is not an integer, and the contours C and D of Theorem 2 and the square bracket convention of Theorem 5 are used, then*

(i) *for real positive x*

$$F_k(x) = -\frac{\sin\pi k}{x}F_k\left(\frac{1}{x}\right)+\frac{2}{\pi}\cos\pi k\,P\int_0^\infty \frac{(xt)^{1+\kappa}}{x^2t^2-1}F_k(t)\,dt; \tag{7.4}$$

(ii)

$$\left[F_k(z)-\frac{ie^{\pi ik}}{z}F_k\left(\frac{1}{z}\right)\right] = \frac{2}{\pi}\cos\pi k\int_C \frac{(zt)^{1+\kappa}}{z^2t^2-1}F_k(t)\,dt,$$

$$\left[F_k(z)+\frac{ie^{-\pi ik}}{z}F_k\left(\frac{1}{z}\right)\right] = \frac{2}{\pi}\cos\pi k\int_D \frac{(zt)^{1+\kappa}}{z^2t^2-1}F_k(t)\,dt;$$

(iii) *if z does not lie on the cut from 0 to $-i\infty$, then*

$$\left[F_k(-z)+\frac{ie^{\pi ik}}{z}F_k\left(-\frac{1}{z}\right)\right]+e^{\pi ik}\left[F_k(z)+\frac{ie^{-\pi ik}}{z}F_k\left(\frac{1}{z}\right)\right]=0.$$

In the cases when $k=p-\frac{1}{2}$ (p an integer) we have $\cos\pi k=0$, $\sin\pi k=(-)^{p-1}$, and (7.3) reduces to an equation of the form

$$F_{p-\frac{1}{2}}(z)=\frac{(-)^p}{z}F_{p-\frac{1}{2}}\left(\frac{1}{z}\right).$$

If we move the line of integration in (7.2) to the left, and use the functional equation of $\zeta(s)$, then we can prove the following results.

THEOREM 9.* *If $R(z)>0$ and p is an integer greater than 1, then*

(i) $$\frac{(-)^p}{4p}B_p+\sum_{n=1}^{\infty}\sigma_{2p-1}(n)e^{-2\pi nz}=\frac{(-)^p}{z^{2p}}\left\{\frac{(-)^p}{4p}B_p+\sum_{n=1}^{\infty}\sigma_{2p-1}(n)e^{-2\pi n/z}\right\};$$

(ii) $$\sum_{n=1}^{\infty}\sigma_1(n)e^{-2\pi nz}=\frac{1}{24z^2}-\frac{1}{4\pi z}+\frac{1}{24}-\frac{1}{z^2}\sum_{n=1}^{\infty}\sigma_1(n)e^{-2\pi n/z};$$

(iii) $$\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2\pi nz}=\frac{\pi}{12z}+\tfrac{1}{2}\log z-\frac{\pi z}{12}+\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2\pi n/z};$$

(iv) $$\sum_{n=1}^{\infty}\sigma_{-2p+1}(n)e^{-2\pi nz}=\tfrac{1}{2}(2\pi)^{2p-1}\sum_{n=1}^{p-1}\frac{(-)^{n+1}z^{2n-1}}{(2n)!\,(2p-2n)!}B_nB_{p-n}+$$
$$+\tfrac{1}{2}B_p\frac{(2\pi)^{2p-1}}{(2p)!}\{z^{-1}+(-)^pz^{2p-1}\}-$$
$$-\tfrac{1}{2}\zeta(2p-1)\{1+(-)^pz^{2p-2}\}+$$
$$+(-)^{p+1}z^{2p-2}\sum_{n=1}^{\infty}\sigma_{-2p+1}(n)e^{-2\pi n/z}.$$

* N. S. Koshliakov, *Messenger of Math.* 58 (1928), 1–23 (19), and S. Wigert, loc. cit. 198.

A REMARK ON UNICURSAL CURVES LYING ON THE GENERAL QUARTIC SURFACE

By B. SEGRE (*Manchester*)

[Received 6 August 1943]

In his Encyclopaedia article on quartic surfaces, W. Fr. Meyer has stated that

*A quartic surface F_4 possesses an invariant J_n, the vanishing of which expresses the necessary and sufficient condition for the existence on F_4 of a unicursal curve of order n.**

This is justified by means of a mere counting of constants, unassisted by any other consideration of the legitimacy of the counting; and nothing is said about the actual formation of the invariants J_n.

I observe now that a justification can be derived for $n \not\equiv 0 \pmod 4$ from Noether's theorem, according to which the only algebraic curves lying on the general quartic surface are its complete intersections with other algebraic surfaces.† But the result quoted is untrue for $n \equiv 0 \pmod 4$, since

The general quartic surface F contains a finite number $c_h > 0$ of unicursal curves of order $4h$ (for $h = 1, 2, 3, \ldots$).

To prove this, I remark that the surfaces of order h cut out on F a linear system $|C|$, of dimension

$$\binom{h+3}{3} - \binom{h-1}{3} - 1 = 2h^2+1$$

and degree $4h^2$. Since F has an effective canonical curve of order zero, $|C|$ is self-adjoint, i.e. its curves cut out canonical sets on the general C. Hence C has the genus $2h^2+1$.

The general curve of $|C|$ is non-singular, and so we must impose $2h^2+1$ conditions in order that it should acquire $2h^2+1$ double points. By a counting of constants, which I shall shortly justify, we see that $|C|$ contains a finite number c_h (> 0) of irreducible curves satisfying such conditions. Each of these curves is of genus zero, and so unicursal.

* W. Fr. Meyer, 'Flächen vierter und höherer Ordnung': *Encycl. Math. Wiss.*, Bd. 3, Teil 2, 2. Halfte B, 1533–1779, § 3.

† Cf. S. Lefschetz, *L'analysis situs et la géométrie algébrique* (Paris, 1924), 108.

The validity of the counting of constants follows from the fact that, if a quartic surface F_4 is intersected by a surface of order h in an irreducible curve possessing $2h^2+1$ double points, then F_4 does not contain an infinity of such curves of order $4h$. For, otherwise, F would contain a continuous system Σ of curves of order $4h$, the generic C_{4h} of which is irreducible and possesses $2h^2+1$ double points. Hence a characteristic set of Σ on C_{4h} (i.e. any limiting position of the intersection set of C_{4h} and another curve of Σ when this curve tends to C_{4h}) would include the $2h^2+1$ double points of C_{4h}, each counted at least twice. But this is impossible, since a characteristic set of C_{4h} consists of $4h^2$ points only.

Noether's theorem shows that the c_h unicursal curves of order $4h$ considered above (for $h = 1, 2, 3,...$), are the *only* unicursal curves lying on the general quartic surface F. The number c_h is not easy to determine for $h \geqslant 2$. For $h = 1$ it obviously equals the number of tritangent planes of F, so that $c_1 = 3200$.*

* Cf. G. Salmon, *Analytic Geometry of Three Dimensions*, vol. ii (5th ed., 1915), § 612.

PROPERTIES OF CERTAIN CUBIC PRIMALS

By J. G. SEMPLE (*London*)

[Received 10 August 1943]

1. LET $V \equiv V_7^3[8]$ denote the determinantal cubic primal whose equation is

$$|x_{ij}| = 0 \qquad (i,j = 1,2,3), \tag{1}$$

and let $D \equiv D_4^6[8]$ denote its double locus.* The well-known duality between properties of V and D can be exhibited in the following scheme :

V	$\tilde{V}$	Γ	Δ	τ	π
$\tilde{D}$	D	γ	δ	T	Π

where the symbols have the following interpretations:

$\tilde{V}$ is the fourfold envelope of tangent primes of V,
$\tilde{D}$ is the sevenfold envelope of tangent primes of D,
Γ, Δ are the sheaves of generating [5] of V,
γ, δ are the sheaves of generating planes of D,
T is a typical tangent [4] of D, join of a pair of planes γ, δ,
τ is a typical contact-solid of V, intersection of spaces Γ, Δ,
Π is a [5], which meets D in a Veronese surface,
π is a plane, vertex of a Veronese envelope† of primes of $\tilde{V}$.

We use, in addition, the symbols w, W as follows:

w is a Veronese surface, of the ∞^8 system on D,
W is a Veronese envelope of primes of $\tilde{V}$.

It may be recalled that (i) spaces Γ, Δ meet D in V_3^3 of γ-planes and δ-planes respectively; (ii) every space T lies on V and is vertex of a web of primes of $\tilde{D}$; (iii) every solid τ meets D in a quadric surface, and every prime of $\tilde{V}$ touches V along such a solid; (iv) the ∞^8 surfaces w are generated by collineations‡ between the sheaves γ and δ; and, dually, (v) the ∞^8 envelopes W are each generated as envelope of primes joining collinearly related spaces Γ, Δ; (vi) each of the ∞^8

* The geometry of D_4^6 and V_7^3 was developed by Segre (**1**) who took them as images of point-pairs and line-pairs respectively of two planes. See also Room (**2**), 100, 101, **139**, **149**, **309**; on pp. 100 and 139 there are some misprints, and the remark on lines **16**, **17** of p. 139 is doubtful.

† This means, of course, the dual in [8] of a Veronese surface.

‡ Planes γ, δ are *collinearly related* if their parameter triads (λ, μ, ν) and (λ', μ', ν') in equations (4) and (5) are so related. Similarly for Γ, Δ.

spaces Π is vertex of a net of primes of $\tilde{D}$; and, dually, (vi) every plane π lies entirely on V.

In his original account of the manifold D, Segre obtained the spaces Π and their dual analogues π, without explicit reference,* however, to the fact that these latter lie on V, or any further note of their properties save that each one of them is associated with a particular space Π in defining an involution (harmonic skew perspective) of S_8 in which V and D are invariant.

We shall want, finally, to refer to spaces—dual to chords, trisecant planes, quadrisecant solids of D—which are intersections of pairs, triads, and tetrads of primes of $\tilde{V}$; and we shall call these *biaxial* [6], *triaxial* [5], *quadraxial* [4] of $\tilde{V}$, respectively.

2. The systems Π and π. From the collineation definition of the surfaces w there follows immediately

THEOREM I. *One and only one space Π passes through four assigned points of D, and two such spaces meet in a trisecant plane of D. Dually, any quadraxial [4] of $\tilde{V}$ contains a unique plane π, and any two such planes lie in a triaxial [5] of $\tilde{V}$.*

To exhibit the further properties of Π and π, it is perhaps simplest to resort to analysis. Let Γ, Δ be given by the respective systems of equations:

$$lx_{i1}+mx_{i2}+nx_{i3}=0 \qquad (i=1,2,3), \tag{2}$$

and

$$l'x_{1j}+m'x_{2j}+n'x_{3j}=0 \qquad (j=1,2,3); \tag{3}$$

and let γ, δ be given by the systems of equations

$$x_{i1}:x_{i2}:x_{i3}=\lambda:\mu:\nu \qquad (i=1,2,3), \tag{4}$$

$$x_{1j}:x_{2j}:x_{3j}=\lambda':\mu':\nu' \qquad (j=1,2,3). \tag{5}$$

The collineation $\lambda':\mu':\nu'=\lambda:\mu:\nu$ defines a typical Veronese surface w_0 lying in the space Π_0 whose equations are

$$x_{23}-x_{32}=x_{31}-x_{13}=x_{12}-x_{21}=0; \tag{6}$$

and similarly the collineation $l':m':n'=l:m:n$ defines the typical Veronese envelope W_0 (of $\tilde{V}$) whose vertex is the plane π_0 given by the equations

$$x_{11}=x_{22}=x_{33}=x_{23}+x_{32}=x_{31}+x_{13}=x_{12}+x_{21}=0. \tag{7}$$

Actually this pair Π_0, π_0 are related in the way described by Segre,

* It seems strange that these planes should have been so little noticed: cf. Room (2), 100.

it being easily verifiable that V and D are invariant in the involution of S_8 of which they are axes.

We observe that π_0 has no intersection with D, so that it cannot lie in any space Γ, Δ, T.

If X_{ij} are the symbols of the vertices of reference, then the spaces Γ, Δ given by (2), (3) meet π_0 in the points P, P' represented symbolically by the forms

$$P \equiv \sum l(X_{23}-X_{32}), \qquad P' \equiv \sum l'(X_{23}-X_{32});$$

and these coincide if $l' : m' : n' = l : m : n$. Furthermore, it may readily be verified that the space T which contains the planes γ, δ given by (4), (5) meets π_0, if at all, in a line;* and this only if $\lambda' : \mu' : \nu' = \lambda : \mu : \nu$.

From this analysis we deduce the following results:

THEOREM II. *In addition to the spaces* Γ, Δ, T, *the primal* V *contains* ∞^8 *planes* π. *Each of these is a simple directrix plane of each of the sheaves,* Γ, Δ, *each prime of* $\tilde{V}$ *which passes through it being the join of spaces* Γ, Δ *which meet it in the same point. The locus of points of contact of tangent spaces* T *of* D *which meet any plane* π *(in lines) is a Veronese surface whose containing space* Π *is associated with* π *in that both are axes of an involution of* S_8 *in which* V *is invariant.*

From his definition of D as the image of point-pairs of two planes, Segre deduced that D projects from any one of the planes π into a repeated cubic symmetroid M_4^3 in [5]. We may add that conic-planes of the double surface of M_4^3 are projections of solids τ (quadric-solids of D) which meet π; tangent-planes of the same surface are projections of spaces T which meet π (in lines); and contact primes of the surface are projections of the ∞^2 primes of $\tilde{V}$ through π.

3. Axial spaces of $\tilde{V}$. Two primes of $\tilde{V}$ meet in a biaxial [6] which touches V along the two planes in which it is met by the contact solids of the primes.

Any such [6] is the join of a unique pair of contact solids τ, just as any chord of D is the intersection of a unique pair of spaces T.

3.1. *Triaxial* [5] *of* $\tilde{V}$. Three primes of $\tilde{V}$ meet in a triaxial [5], Θ say, which touches V along three lines, namely those in which it is

* The analysis shows that each line of π_0 is the intersection of this plane with a unique space T. In fact, if $\lambda' : \mu' : \nu' = \lambda : \mu : \nu$, then T meets π_0 in the line whose equation, in this plane, is $\sum \lambda(x_{23}-x_{32}) = 0$.

met by the contact solids of the primes. From the representation of D (on a [4] by means of quadrics through two skew lines), or by duality, it appears that Θ meets D in a hexagon of lines whose diagonals are the three lines of contact referred to. It follows that Θ meets V in what we may call a *Perazzo* cubic primal* V_4^3, possessing nine double lines along the sides and diagonals of a skew hexagon. Conversely, any [5] which meets D in a skew hexagon of lines (lying in alternate distinct planes γ, δ) necessarily touches V along the three diagonals of this hexagon. Since a generic [6] meets D in a sextic Del Pezzo surface containing just such a hexagon (forming a special prime section of the surface), it follows that any such [6] contains a unique triaxial [5].

It follows from this, by duality, that a unique trisecant plane of D passes through an arbitrary line.

A convenient example of a triaxial space is that given by

$$x_{11} = x_{22} = x_{33} = 0. \tag{8}$$

This meets V in the Perazzo cubic primal whose equation is

$$x_{23}x_{31}x_{12} = -x_{32}x_{13}x_{21}. \tag{9}$$

3.2. *Quadraxial* [4] *of* $\tilde{V}$. Four primes of $\tilde{V}$ meet in a quadraxial [4] which touches V in the four points in which it is met by the contact solids of the primes. It is therefore a quadritangent [4] of V; and conversely any quadritangent [4] of V is obviously a quadraxial [4] of $\tilde{V}$.

By Theorem I any quadraxial [4] of $\tilde{V}$ contains a unique plane π. But, conversely, any [4] through a plane π is quadraxial; for it lies necessarily in four primes of the quartic envelope W of primes of $\tilde{V}$ through π, and it touches V at the points of contact of these primes in π. Hence we have

THEOREM III. *A* [4] *is quadritangent to* V (*meeting it in a Segre* 10*-nodal* V_3^3) *if and only if it contains a plane* π *of* V. *Its four points of contact then lie in this plane.*

4. Double projection of Del Pezzo F_2^6. Since D projects into a repeated cubic symmetroid from any π-plane of V, it follows that the Del Pezzo surface F_2^6 in which V is met by an S_6 will project into a repeated 4-nodal cubic surface from any π-plane of V which lies

* This is the V_4^3, studied in Perazzo (**3**), representable on S_4 by means of quadrics through a fixed skew quadrilateral and one further fixed point.

in S_6. It may be of interest to locate such π-planes in S_6 and to interpret the projection.

Planes π which lie in a secundum are dual to spaces Π which pass through a line l. Now, by § 3.1, there is a unique trisecant plane of D through l, which must be common to every space Π through l, meeting D in a Veronese surface. It follows then that all π-planes of V which lie in a secundum S_6 lie in the unique triaxial [5], Θ, which is contained in this S_6. Now Θ meets V in a Perazzo cubic primal V_4^3 whose nine double lines form a skew hexagon on D, together with the diagonals of this hexagon; one and only one system of generating planes of V_4^3 meet the diagonals but not the sides of this hexagon, and *these are therefore the π-planes of V which lie in S_6.*

If we represent F_2^6 parametrically in the form

$$\frac{x_0}{xyz}=\frac{x_1}{y^2z}=\frac{x_1'}{yz^2}=\frac{x_2}{z^2x}=\frac{x_2'}{x^2z}=\frac{x_3}{x^2y}=\frac{x_3'}{xy^2}, \tag{10}$$

then its chord-locus V_5^3 has equation

$$\begin{vmatrix} x_0 & x_1 & x_1' \\ x_2' & x_0 & x_2 \\ x_3 & x_3' & x_0 \end{vmatrix} = 0; \tag{11}$$

the prime Θ whose equation is $x_0 = 0$ meets F_2^6 in the linear hexagon $X_1X_1'X_2X_2'X_3X_3'$ and meets V_5^3 in the Perazzo primal whose equation (in Θ) is

$$x_1x_2x_3 = -x_1'x_2'x_3'; \tag{12}$$

and the required system of planes π, from any one of which F_2^6 projects doubly, is that given by the equations

$$\left.\begin{aligned} x_2 = \lambda x_3', \qquad x_3 = \mu x_1', \qquad x_1 = \nu x_2' \\ x_0 = 0, \qquad \lambda\mu\nu = -1 \end{aligned}\right\}. \tag{13}$$

If π_0 is the plane of this system given by $\lambda = \mu = \nu = -1$, then the web of plane cubics which represent sections of F_2^6 by primes through π_0 has equation

$$\lambda_0 xyz+\lambda_1 x(y^2+z^2)+\lambda_2 y(z^2+x^2)+\lambda_3 z(x^2+y^2) = 0. \tag{14}$$

This is clearly an *involutory web** since curves of it which pass through (x, y, z) also pass through (x^{-1}, y^{-1}, z^{-1}). A fairly simple

* Cf. Bronowski (4), 14 (1943), 11.

elimination shows that the projection of F_2^6 from π_0 is the 4-nodal cubic surface whose equation is

$$T^3+2XYZ = T(X^2+Y^2+Z^2), \tag{15}$$

where $X : Y : Z : T = x(y^2+z^2) : y(z^2+x^2) : z(x^2+y^2) : 2xyz$.

5. Veneroni's Axial V_4^3. In a long paper on determinantal cubic primals* of S_5, Veneroni discovered the existence of a special type—what we may call the *axial* V_4^3—such that, although its double elliptic sextic curve D_1^6 is perfectly general, yet it possesses a simple directrix plane—its *axis*—met by every plane of both its generating systems.† It seems worth while to point out that, obviously, *an axial V_4^3 is simply a section of V by a* [5] *which contains a plane π*; and most of its peculiar properties thereby become apparent. Thus, for example, the generic V_4^3—section of V by a generic S_5—has ∞^2 tritangent primes, namely those in which it is met by the triaxial [5] which lie in [6] through S_5; but if S_5 contains a plane π, then V_4^3 possesses instead a net of quadritangent primes‡ through π.

Besides its two sheafs of generating planes, which we may conveniently call Γ-*planes* and Δ-*planes*, the generic V_4^3 (section of V by S_5) contains a further simply infinite family of T-*planes*, sections of T-spaces of V which happen to meet S_5 in planes. If S_5 contains a plane π, then Theorem II tells us that π is met in a line by the tangent [4] of D at every point of a certain Veronese surface w; D projects doubly from π into the chord-locus M_4^3 of w, and D_1^6 into a plane section D_1^3 of M_4^3; two tangent planes of w (projections of T-spaces which meet π in lines) pass through each point of D_1^3; and hence every point of D_1^3 is the projection of two T-planes of V_4^3 which meet π in lines. Also, as the point varies on D_1^6, the associated points of contact on w describe an elliptic sextic curve and the associated lines on π generate a cubic envelope. Hence

* Veneroni (**5**), § 16 and sequel.

† This result of Veneroni's was, unfortunately, misapprehended by Segre, whose note in *Encyk. d. Math Wiss.* III C 7, 953 f.n. (563), assigns multiplicities 3 and 2 respectively, on V_4^3, to the axial plane and a certain sextic scroll to which we refer later. Both are simple on V_4^3, their multiplicities, to which Segre refers, being relative to a certain other locus used by Veneroni.

‡ In the elliptic parametric representation of D_1^6, later referred to, the tritangent primes of the general V_4^3 and the quadritangent primes of the axial V_4^3 are those which meet D_1^6 in sets of six points whose parameters are of the form

$$\alpha\pm k, \qquad \beta\pm k, \qquad \gamma\pm k,$$

where $\alpha+\beta+\gamma \equiv 0$.

The T*-planes of an axial primal* V_4^3 *meet the axial plane of this primal in the lines of a cubic envelope.**

The curve D_1^6 (and the axial primal V_4^3) are invariant under the involution of S_5 whose axes are the plane π and the plane in which S_5 meets the associated space Π.

5.1. *Linear pencil of* V_4^3. The V_4^3 of S_5 which have an *assigned* double curve D_1^6 form a linear pencil whose base is the chord-locus of D_1^6; and Veneroni showed that this pencil contains—besides four cubic symmetroids—six axial primals. As Veneroni did not use the elliptic parametric representation of D_1^6, it seems worth while to indicate briefly the very simple way in which the six axial primals are exhibited in such a representation. We suppose as usual that six co-prime points of D_1^6 have a parameter-sum congruent to zero, relative to the periods ω, ω'.

The two sheafs of generating planes Γ, Δ of any V_4^3 of the pencil are given—as trisecant planes of D_1^6—by a pair of congruences of the forms†

$$u_1+u_2+u_3 \equiv \pm k. \tag{16}$$

The T-planes of V_4^3—bisecant planes of D_1^6—cut D_1^6 in pairs of points P, P' of the algebraic series γ_2^1 given by

$$u-u' \equiv 2k, \tag{17}$$

any one of them being met in a line by all Γ-planes through P and by all Δ-planes through P'. The axial primals of the pencil are those for which γ_2^1 reduces to one of the *three elliptic involutions* on D_1^6, so that *two* T-planes of V_4^3 now pass through each pair P, P'. The six pairs of values of k which characterize the axial primals of the pencil are accordingly

$$\begin{gathered}\pm\tfrac{1}{4}\omega, \quad \pm\tfrac{1}{4}(\omega+2\omega'), \quad \pm\tfrac{1}{4}\omega', \quad \pm\tfrac{1}{4}(2\omega+\omega'), \\ \pm\tfrac{1}{4}(\omega+\omega'), \quad \pm\tfrac{1}{4}(\omega-\omega').\end{gathered} \tag{18}$$

If τ is the involution given by $u' \equiv u+\frac{1}{2}\omega$, then the first two of the axial primals are those associated with τ, and so on.

The joins of pairs of points P, P' which correspond in τ generate an *elliptic sextic scroll* which is met in cubic directrix curves‡ by two planes π_1, π_2, the axial planes respectively of the two associated

* Veneroni (5), § 29.

† The four values $0, \frac{1}{2}\omega, \frac{1}{2}\omega', \frac{1}{2}(\omega+\omega')$ of k yield, as is well known, the four cubic symmetroids of the pencil, for which the systems Γ, Δ coalesce.

‡ Cf. Segre (6), §§ 11, 13.

axial primals; and these primals, as well as D_1^6 itself, are invariant in the involution of S_5 whose axes are π_1, π_2.

5.2. *S_4-representation of axial V_4^3.* The distinction between the general and the axial V_4^3 is clearly exhibited in their S_4-representations,* obtained by projection from a point of D_1^6. These representations are by means of cubic primals of S_4 passing through two fixed cubic scrolls which intersect in an elliptic D_1^5. If V_4^3 is axial, then the directrices of the scrolls meet D_1^5 in the same point and so lie in a plane; and it is this plane which images the axial plane of the primal.

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* An alternative quartic representation of V_4^3 exists in which the two sheafs of planes Γ, Δ are more simply represented by nets of planes through two lines g, g' respectively. This results from transforming the representation given above by means of the Cremona quadric transformation based on D_1^2; the two cubic scrolls transform into g, g'.

A PROPERTY OF CERTAIN DIFFERENTIAL EQUATIONS

By W. W. SAWYER (*Manchester*)

[Received 20 August 1943]

Statement of results

LET E_0 be the differential equation of the second order

$$a_0(x)y''+b_0(x)y'+c_0(x)y = 0,$$

with solutions $u(x)$ and $v(x)$. Let E_n be the equation

$$a_n(x)y''+b_n(x)y'+c_n(x)y = 0,$$

the solutions of which are $u^{(n)}(x)$ and $v^{(n)}(x)$, the nth differential coefficients of the solutions of E_0. We suppose the equations E_n to be of Fuchsian type, with $a_n(x)$, $b_n(x)$, $c_n(x)$ polynomials in x.

If $u(x)$ has a singularity at $x = \xi$, the same will be true of $u^{(n)}(x)$. Thus ξ will be a singularity of all the equations E_n, and $a_n(\xi) = 0$ for all n.

There may also be points for which $a_n(x) = 0$, although $u^{(n)}(x)$ and $v^{(n)}(x)$ are both regular there. Such points are called *apparent singularities*† *of* E_n. In general, an apparent singularity of E_n will not be an apparent singularity of E_{n+1}.

THEOREM I. *If the degree of* $a_n(x)$, *regarded as a function of* x, *is the same for all* n, *the equation* E_0 *has at most three singularities* (*one of which must be at infinity*), *together with a finite number* ρ *of apparent singularities.*

THEOREM II. *The apparent singularities of* E_n *are the roots of an equation*

$$\Phi_n(x) \equiv x^\rho+\alpha_{1,n}x^{\rho-1}+\alpha_{2,n}x^{\rho-2}+\ldots+\alpha_{\rho,n} = 0 \qquad (1)$$

in which $\alpha_{s,n}$ is a polynomial in n of degree s.

Proof of Theorem I

Let E_n have N singularities and R_n apparent singularities. Let σ_s, τ_s be the indices at the sth singularity, It is known that R_n differs from $-\sum_{s=1}^{N}(\sigma_s+\tau_s)$ by a fixed number.‡

† See E. L. Ince, *Ordinary Differential Equations*, Section 16.4.

‡ Klein, *Hypergeometrische Funktion* (Springer, 1933), § 24.

In general, E_{n+1} will have more apparent singularities than E_n, since differentiation of y reduces the index σ_s to σ_s-1. In general, $R_{n+1}-R_n = 2N$. The following exceptions exist:

(i) if ∞ is a singularity, differentiation *increases* each index for ∞ by unity;

(ii) if at any singularity one index is zero, differentiation usually leaves this index unaltered; if, in the neighbourhood of $x = \xi$, a solution is represented by a power series $P(x-\xi)$ in which some coefficients are zero, successive differentiation will cause the index to fluctuate, with the lower bound zero.

Thus a general singularity contributes $+2$ to $R_{n+1}-R_n$; a singularity at ∞ contributes -2; a singularity with one zero index contributes (in general) $+1$.

Apart from trivial cases, R_n can be kept constant in only two ways: (i) there may be a singularity at infinity and two finite singularities at each of which one index is zero; (ii) the equation may have one finite singularity and a singularity at infinity.

We shall show later that case (ii) may be regarded as a particular instance of case (i), which arises when the two singularities coalesce.

Case (i). Without loss of generality, we may suppose the singularities to be 0, ∞, 1. Let the indices for E_n be $(p_n, 0)$ at $x = 0$, $(q_n, 0)$ at $x = 1$, (β_n, β_n^*) at $x = \infty$.

Let S_r be the rth apparent singularity. Then E_n must have the form

$$y''+y'\left\{\frac{1-p_n}{x}+\frac{1-q_n}{x-1}-\sum_r\frac{1}{x-S_r}\right\}+y\,\frac{\beta_n\beta_n^*P_n(x)}{x(x-1)\prod(x-S_r)}=0 \quad (2)$$

in which $P_n(x)$ is a polynomial of degree ρ, where

$$\rho = 1-p_n-q_n-\beta_n-\beta_n^*$$

is the number of apparent singularities. It may be seen by considering the behaviour of y near $x = \infty$ that the coefficient of x^ρ in $P_n(x)$ must be unity. Let $\prod(x-S_r) = \Phi_n(x)$, so that $\Phi_n(x) = 0$ gives the apparent singularities of E_n.

Let ξ be a root of $P_n(x) = 0$. When $x = \xi$, y_n''/y_n' has a determinate value. That is to say, the differential equation for y_n', which is E_{n+1}, has an apparent singularity at $x = \xi$. Hence any root of $P_n(x) = 0$ is a root of $\Phi_{n+1}(x) = 0$. As both $P_n(x)$ and $\Phi_{n+1}(x)$ have the highest term x^ρ, it follows that $P_n(x) \equiv \Phi_{n+1}(x)$.

Thus the equation E_n is

$$y''+y'\left\{\frac{1-p_n}{x}+\frac{1-q_n}{x-1}-\frac{\Phi_n'}{\Phi_n}\right\}+y\frac{\beta_n\beta_n^*\Phi_{n+1}(x)}{x(x-1)\Phi_n(x)}=0,$$

in which $p_n=p_0-n$, $q_n=q_0-n$, $\beta_n=\beta_0+n$, $\beta_n^*=\beta_0^*+n$. E_n may also be written

$$y''+y'\left\{\frac{F_n}{G}-\frac{\Phi_n'}{\Phi_n}\right\}+\frac{yk_n\Phi_{n+1}}{G\Phi_n}=0 \tag{2a}$$

where

$$G=x(x-1), \qquad F_n=(1-p_n)(x-1)+(1-q_n)x \tag{2b}$$
$$k_n=\beta_n\beta_n^*.$$

Case (ii). Confluence of the singularities in equation (2) may be produced by writing

$$\left.\begin{aligned} x&=z/\epsilon \\ S_r&=Z_r/\epsilon \quad (r=1,2,\ldots,\rho)\end{aligned}\right\}$$

and allowing ϵ to tend to zero, the quantities Z_r remaining constant. The equation thus becomes

$$\frac{d^2y}{dz^2}+\frac{dy}{dz}\left\{\frac{\kappa_n}{z}-\sum_r\frac{1}{z-Z_r}\right\}+y\frac{k_nM_n(z)}{z^2\pi(z-Z_r)}=0, \tag{2c}$$

where

$$\kappa_n=2-p_n-q_n, \qquad k_n=\beta_n\beta_n^*,$$
$$M_n(z)=\lim_{\epsilon\to 0}\epsilon^\rho P_n\left(\frac{z}{\epsilon}\right).$$

The equation (2c) has a singularity at $z=0$ with arbitrary indices, and apparent singularities at the ρ arbitrary points $Z_1,\ldots,Z_\rho$. It is thus the most general equation of the type considered in case (ii).

In general, the solutions of this equation are of the form $z^\alpha A(z)$, in which $A(z)$ is a polynomial. If the indices at $z=0$ differ by an integer or zero, there may be a solution of the form

$$z^\alpha A_1(z)\log z+z^\beta A_2(z),$$

where $A_1(z)$ and $A_2(z)$ are polynomials.

Proof of Theorem II

If the equation $$y''+By'+Cy=0 \tag{3}$$

has solutions u and v, the equation with solutions u' and v' is

$$y''+y'\left\{B-\frac{C'}{C}\right\}+yC\left\{1+\left(\frac{B}{C}\right)'\right\}=0. \tag{4}$$

The equation E_n is derived from E_{n-1} in the same way as (4) is derived from (3). Writing E_{n-1} and E_n in the form given by (2a), it is found that the coefficient of y' in E_n automatically takes the correct value. The coefficient of y yields the relation

$$\Delta\left\{\frac{k_{n-1}\Phi_n}{\Phi_{n-1}}\right\} = \frac{\Phi_n}{\Phi_{n-1}}\left\{\frac{F_{n-1}\Phi_{n-1}-G\Phi'_{n-1}}{\Phi_n}\right\}',$$

in which Δ is the finite difference operator for the variable n: that is, $\Delta f_n \equiv f_{n+1}-f_n$ for any f_n. The relation above reduces to

$$k_n\Phi_{n-1}\Phi_{n+1}-k_{n-1}\Phi_n^2 = \Phi_n Q'_n-\Phi'_n Q_n \qquad (5)$$

where $$Q_n \equiv F_{n-1}\Phi_{n-1}-G\Phi'_{n-1}.$$

If in equation (5) we substitute the expression for $\Phi_n(x)$ given in equation (1), and equate coefficients of $x^{2\rho-s}$, we obtain

$$\sum_{t=0}^{s}\left\{\begin{array}{l} k_n\,\alpha_{t,n-1}\,\alpha_{s-t,n+1}-k_{n-1}\,\alpha_{t,n}\,\alpha_{s-t,n}+ \\ \quad +(2t-s-1)(\Delta k_{n-1}+t)\alpha_{s-t,n}\,\alpha_{t,n-1}+ \\ \qquad +(2t-s-1)(p_0-n+\rho-t+1)\alpha_{s-t,n}\,\alpha_{t-1,n-1} \end{array}\right\} = 0 \qquad (6)$$

in which $\alpha_{0,n} = 1$ and $\alpha_{t,n}$ is to be regarded as zero if t does not belong to the set 0, 1, 2,..., ρ.

Equation (6) takes a particularly simple form for $s = 1$, becoming merely

$$k_n\{\alpha_{1,n+1}-2\alpha_{1,n}+\alpha_{1,n-1}\} = 0.$$

That is, $k_n\,\Delta^2\alpha_{1,n-1} = 0$, so that $\alpha_{1,n}$ must be linear in n.

If s is taken in turn equal to 1, 2, 3,..., the quantities $\alpha_{s,n-1}$, $\alpha_{s,n}$, and $\alpha_{s,n+1}$ appear for the first time in the sth equation: all other terms in this equation contain α-symbols with the first index less than s. The sth equation may be regarded as a difference-equation for $\alpha_{s,n}$, and it is thus possible to find in turn $\alpha_{1,n}$, $\alpha_{2,n}$, etc., two arbitrary constants appearing at each stage, as s goes from 1 to ρ. The equations with $s > \rho$ are not needed for the proof of the theorem.

It is thus natural to establish a proof by induction, showing that, if $\alpha_{r,n}$ is of degree r for $r < s$, $\alpha_{s,n}$ is of degree s. We know that $\alpha_{1,n}$ is of degree 1.

For any $s \leqslant \rho$, the terms in equation (6) which contain α_s arise from $t = 0$ and $t = s$, and may be written as

$$k_n\,\Delta^2\alpha_{s,n-1}-(s-1)\Delta k_{n-1}\,\Delta\alpha_{s,n-1}+s(s-1)\alpha_{s,n-1}. \qquad (7)$$

The expression (7) has the property that its $(s-1)$th difference equals $k_{n+s-1}\Delta^{s+1}\alpha_{s,n-1}$, since k_n is quadratic in n, and $\Delta^2 k_n = 2$. Hence the desired result will follow if we can show that the terms of (6), not contained in (7), are of degree $(s-2)$ at most.

The proof which follows is neither elegant nor enlightening. It merely establishes the truth of the result stated, leaving for later investigation the task of finding the inner significance and the simplest proof of the whole theorem.

The terms to be considered fall into two parts:

$$\sum_{t=1}^{t=s-1}\{k_n\alpha_{t,n-1}\alpha_{s-t,n+1}-k_{n-1}\alpha_{t,n}\alpha_{s-t,n}+ \\ +(2t-s-1)(\Delta k_{n-1}+t)\alpha_{s-t,n}\alpha_{t,n-1}\}, \quad (8a)$$

$$\sum_{t=1}^{s}\{(2t-1-s)(p_0-n+\rho-t+1)\alpha_{s-t,n}\alpha_{t-1,n-1}\}. \quad (8b)$$

In (8b) we consider together the terms with $t = q$ and those with $t = s+1-q$. For these values of t the first bracket $2t-1-s$ takes equal and opposite values. Removing this constant factor $(2q-1-s)$ there remains an expression which has degree $(s-2)$ in virtue of the following lemma.

LEMMA 1. *If U_n and V_n are polynomials of degrees u and v respectively, and L_n is linear in n, then $L_{n+u-v}U_{n-1}V_n - L_n U_n V_{n-1}$ is of degree $(u+v-1)$ at most.*

The proof of the lemma is trivial. For the application above, let

$$U_n = \alpha_{q-1,n}, \qquad V_n = \alpha_{s-q,n},$$

so that $u+v-1 = s-2$. Hence (8b) is of degree $(s-2)$.

In (8a) we consider together the terms for which $t = q$ and those with $t = s-q = j$ say.

These terms may be divided into the following parts:

$$R_1 \quad \alpha_{q,n-1}\{k_n\Delta^2\alpha_{j,n-1}-(j-1)\Delta k_{n-1}\Delta\alpha_{j,n-1}+j(j-1)\alpha_{j,n-1}\},$$

$$R_2 = \alpha_{j,n-1}\{k_n\Delta^2\alpha_{q,n-1}-(q-1)\Delta k_{n-1}\Delta\alpha_{q,n-1}+q(q-1)\alpha_{q,n-1}\},$$

$$R_3 = -2k_{n-1}\Delta\alpha_{q,n-1}\Delta\alpha_{j,n-1}+ \\ +(\Delta k_{n-1}-1)(q\alpha_{q,n-1}\Delta\alpha_{j,n-1}+j\alpha_{j,n-1}\Delta\alpha_{q,n-1})- \\ -2qj\alpha_{q,n-1}\alpha_{j,n-1},$$

$$R_4 = (q-)\{q\alpha_{q,n-1}\Delta\alpha_{j,n-1}-j\alpha_{j,n-1}\Delta\alpha_{q,n-1}\}.$$

In R_1, the bracketed expression has the same form as expression (7). Its $(j-1)$th difference is $k_{n+j-1}\Delta^{j+1}\alpha_{j,n-1} = 0$ since $\alpha_{j,n}$ is of degree j. Hence the bracketed expression is of degree $(j-2)$ at most. Hence R_1 is of degree $q+j-2 = s-2$, at most.

Similarly, R_2 is of degree $(s-2)$ at most.

LEMMA 2. *Let U, V, Y, Z, Q be polynomials in n such that*

$$U = n^{\lambda}+\ldots, \qquad V = n^{\lambda-1}+\ldots,$$
$$Y = n^{\mu}+\ldots, \qquad Z = n^{\mu-1}+\ldots,$$
$$Q = n^{2}+\ldots$$

where the highest term of each expression is given. Then

$$2UY-(\Delta Q-1)(UZ+YV)+2QVZ \tag{9}$$

is of degree $(\lambda+\mu-2)$ at most.

Proof. Let $$Q = (n-e)(n-f).$$

Then $$\Delta Q-1 = 2n-e-f = (n-e)+(n-f).$$

Hence the expression (9) equals

$$\{U-(n-e)V\}\{Y-(n-f)Z\}+\{U-(n-f)V\}\{Y-(n-e)Z\}$$

which is obviously of degree $\lambda+\mu-2$.

Now, if we write

$$Q = k_{n-1}, \qquad U = \alpha_{q,n-1}, \qquad Y = \alpha_{j,n-1},$$
$$V = \frac{1}{q}\Delta\alpha_{q,n-1}, \qquad Z = \frac{1}{j}\Delta\alpha_{j,n-1},$$

the conditions of Lemma 2 are satisfied, with $\lambda = q$ and $\mu = j$. Substituting in (9), and multiplying by $-qj$, we see that R_3 has degree not higher than $q+j-2 = s-2$.

The expression R_4 is obviously of degree $q+j-2$, since the coefficient of the highest power, n^{q+j-1} is zero.

The proof of Theorem II is thus complete.

AN EXTENSION OF THE STURM-LIOUVILLE EXPANSION

By E. C. TITCHMARSH (*Oxford*)

[Received 5 November 1943]

1. LET L denote the linear matrix-operator

$$L \equiv \begin{bmatrix} p(x), & q(x)+\dfrac{d}{dx} \\ q(x)-\dfrac{d}{dx}, & r(x) \end{bmatrix},$$

and $f(x)$, $g(x)$, $\psi(x)$, $\Phi(x,w)$ functions with two components, $f_1(x)$, $f_2(x)$, etc. If f is represented by the two-rowed matrix

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

and Lf is formed by the rule of matrix multiplication, then

$$Lf = \begin{bmatrix} f_2'+pf_1+qf_2 \\ -f_1'+qf_1+rf_2 \end{bmatrix}.$$

The equation

$$(L-w)f = 0 \tag{1.1}$$

is equivalent to the two differential equations

$$\left.\begin{aligned} f_2'+pf_1+qf_2-wf_1 &= 0 \\ -f_1'+qf_1+rf_2-wf_2 &= 0 \end{aligned}\right\}. \tag{1.2}$$

This linear operator will give rise to a formula for the expansion of an arbitrary function in terms of eigenfunctions. The particular case where

$$p(x) = -\gamma-\frac{\alpha}{x}, \qquad q(x) = \frac{j}{x}, \qquad r(x) = \gamma-\frac{\alpha}{x},$$

and the interval is $(0,\infty)$ was considered in a previous paper.* This, however, is a 'singular' case. The object of the present paper is to consider the case where x lies in a finite interval (a,b), in which $p(x)$, $q(x)$, $r(x)$ and their first derivatives are continuous. We prove that in this case the expansion converges under substantially the same conditions as an ordinary Fourier series.

If in (1.2) we replace p, r, w and f_2 by

$$\epsilon p+1/\epsilon, \quad \epsilon r-1/\epsilon, \quad \epsilon w+1/\epsilon, \quad \epsilon f_2$$

* E. C. Titchmarsh, *Quart. J. of Math.* (Oxford), 13 (1942), 1–10.

respectively, we obtain

$$\left(\frac{d}{dx}-q\right)f_1 = -2f_2+\epsilon^2(r-w)f_2,$$

$$-\left(\frac{d}{dx}+q\right)f_2 = (p-w)f_1.$$

Hence
$$\left(\frac{d}{dx}+q\right)\left(\frac{d}{dx}-q\right)f_1 = 2(p-w)f_1+\epsilon^2\left(\frac{d}{dx}+q\right)(r-w)f_2.$$

Making $\epsilon \to 0$ $$\left(\frac{d^2}{dx^2}-q^2-q'-2p+2w\right)f_1 = 0.$$

This is of the form which gives rise to the ordinary Sturm-Liouville expansion.

2. We start from the following existence theorem.

Let (a, b) be a finite interval, in which $p(x)$, $q(x)$, and $r(x)$ are continuous functions of x. Then (1.2) *has a solution $f(x) = f(x, w)$ such that*

$$f_1(a) = \sin\alpha, \qquad f_2(a) = -\cos\alpha,$$

where α is any given constant. The components of this solution are integral functions of w.

This follows from the classical method of successive approximations. Let

$$f_1^{(0)}(x) = \sin\alpha, \qquad f_2^{(0)}(x) = -\cos\alpha$$

and

$$f_1^{(n)}(x) = \int_a^x \{q(y)f_1^{(n-1)}(y)+r(y)f_2^{(n-1)}(y)-wf_2^{(n-1)}(y)\}\,dy + \sin\alpha,$$

$$-f_2^{(n)}(x) = \int_a^x \{p(y)f_1^{(n-1)}(y)+q(y)f_2^{(n-1)}(y)-wf_1^{(n-1)}(y)\}\,dy + \cos\alpha,$$

for $n = 1, 2, \ldots$. Let

$$u^{(n)}(x) = f^{(n)}(x)-f^{(n-1)}(x) \quad (n = 1, 2, \ldots).$$

Then

$$u_1^{(n)}(x) = \int_a^x (qu_1^{(n-1)}+ru_1^{(n-1)}-wu_2^{(n-1)})\,dy \quad (n = 2, 3, \ldots).$$

Let $$|p(x)|+|q(x)|+|r(x)| \leqslant M$$

and $|u_1^{(1)}| \leqslant K$, $|u_2^{(1)}| \leqslant K$ for $a \leqslant x \leqslant b$, $|w| \leqslant N$. Then

$$|u_1^{(2)}(x)| \leqslant \int_a^x (M+N)K\,dy = (x-a)(M+N)K,$$

and generally

$$|u_1^{(n)}(x)| \leqslant \frac{(x-a)^{n-1}}{(n-1)!}(M+N)^{n-1}K,$$

and similarly for u_2. Hence the series

$$f_1(x) = \sin\alpha + \sum_{n=1}^{\infty} u_1^{(n)}(x), \qquad f_2(x) = -\cos\alpha + \sum_{n=1}^{\infty} u_2^{(n)}(x)$$

converge uniformly with respect to w for $|w| \leqslant N$, and it is easily verified that they satisfy the required conditions. This proves the theorem.

3. Let $\psi(x)$ be an arbitrary integrable function of x in (a, b), i.e. let its components $\psi_1(x)$, $\psi_2(x)$ be arbitrary integrable functions. As in the case of the ordinary Sturm-Liouville expansion, we require the function $\Phi(x, w)$ which satisfies the differential equation

$$(L-w)\Phi(x, w) = -\psi(x), \tag{3.1}$$

and the boundary conditions

$$\left.\begin{aligned} \Phi_1(a, w)\cos\alpha + \Phi_2(a, w)\sin\alpha &= 0 \\ \Phi_1(b, w)\cos\beta + \Phi_2(b, w)\sin\beta &= 0 \end{aligned}\right\}. \tag{3.2}$$

Define $f(x, w)$ as in § 2, and let $g(x, w)$ be the corresponding function with a, α replaced by b, β. Let

$$\phi(w) = g_1(x, w)f_2(x, w) - f_1(x, w)g_2(x, w). \tag{3.3}$$

It follows from (1.2) that $\partial\phi/\partial x = 0$, so that $\phi(w)$ is independent of x. Let

$$\Phi(x, w) = \frac{g(x, w)}{\phi(w)} \int_a^x f(y, w)\psi(y)\, dy + \frac{f(x, w)}{\phi(w)} \int_x^b g(y, w)\psi(y)\, dy, \tag{3.4}$$

where $f\psi$ denotes $f_1\psi_1 + f_2\psi_2$, etc. It is easily verified that this is the solution of (3.1), (3.2).

Suppose that the only zeros of $\phi(w)$ are simple zeros w_1, w_2,... on the real axis. Then

$$f_1(x, w_n)g_2(x, w_n) = g_1(x, w_n)f_2(x, w_n). \tag{3.5}$$

Also $$f_1 g_1' - g_1 f_1' = (r-w)(f_1 g_2 - g_1 f_2).$$

Hence $$f_1(x, w_n)g_1'(x, w_n) - g_1(x, w_n)f_1'(x, w_n) = 0. \tag{3.6}$$

Hence there are constants k_n such that

$$g_1(x, w_n) = k_n f_1(x, w_n), \tag{3.7}$$

and then by (3.5) $$g_2(x, w_n) = k_n f_2(x, w_n). \tag{3.8}$$

Hence, at $w = w_n$, $\Phi(x, w)$ has the residue

$$\frac{k_n}{\phi'(w_n)} f(x, w_n) \int_a^b f(y, w_n)\psi(y)\, dy.$$

Assuming that

$$\psi(x) = \lim_{N\to\infty} \frac{1}{2\pi i} \int_{|w|=N} \Phi(x, w)\, dw$$

we should then obtain the expansion

$$\psi(x) = \sum_{n=1}^{\infty} \frac{k_n}{\phi'(w_n)} f(x, w_n) \int_a^b f(y, w_n)\psi(y)\, dy. \tag{3.9}$$

This is equivalent to two formulae, of which the first is

$$\psi_1(x) = \sum_{n=1}^{\infty} \frac{k_n}{\phi'(w_n)} f_1(x, w_n) \int_a^b \{f_1(y, w_n)\psi_1(y) + f_2(y, w_n)\psi_2(y)\}\, dy.$$

Since ψ_1 and ψ_2 are independent, this is equivalent to

$$\psi_1(x) = \sum_{n=1}^{\infty} \frac{k_n}{\phi'(w_n)} f_1(x, w_n) \int_a^b f_1(y, w_n)\psi_1(y)\, dy \tag{3.10}$$

and

$$0 = \sum_{n=1}^{\infty} \frac{k_n}{\phi'(w_n)} f_1(x, w_n) \int_a^b f_2(y, w_n)\psi_2(y)\, dy. \tag{3.11}$$

Here (3.10) has exactly the same appearance as if the $f_1(x, w_n)$ were an orthogonal set, though it is easy to see from examples that this is not necessarily so.

The details of the expansion are easily worked out in the 'Fourier' case $p(x) = c$ (a constant), $q(x) = 0$, $r(x) = -c$.

4. Orthogonal property of the expansion

For any functions f, F,

$$FLf = F_1 f_2' - f_1' F_2 + pF_1 f_1 + q(F_1 f_2 + f_1 F_2) + rF_2 f_2.$$

Hence

$$fLF - FLf = \frac{d}{dx}(f_1 F_2 - F_1 f_2). \tag{4.1}$$

Hence

$$\int_a^b fLF\, dx = \int_a^b FLf\, dx,$$

provided that $f_1F_2-F_1f_2$ vanishes at $x=a$ and $x=b$. Now

$$f_1(a,w)\cos\alpha+f_2(a,w)\sin\alpha=0 \tag{4.2}$$

for all w; and

$$f_1(b,w)\cos\beta+f_2(b,w)\sin\beta=\phi(w) \tag{4.3}$$

which vanishes for $w=w_m$, $w=w_n$. Hence

$$\int_a^b f(x,w_m)Lf(x,w_n)\,dx=\int_a^b f(x,w_n)Lf(x,w_m)\,dx. \tag{4.4}$$

Hence, by (1.1),

$$(w_m-w_n)\int_a^b f(x,w_m)f(x,w_n)\,dx=0. \tag{4.5}$$

Hence, if $w_m\neq w_n$,

$$\int_a^b f(x,w_m)f(x,w_n)\,dx=0, \tag{4.6}$$

the orthogonal property. As in the Sturm-Liouville case, (4.5) also shows that all the zeros of $\phi(w)$ are real.

Again, by (4.1), (1.1), and (4.2), for any w, w',

$$\begin{aligned}(w-w')\int_a^b f(x,w)f(x,w')\,dx&=f_1(b,w')f_2(b,w)-f_2(b,w')f_1(b,w)\\&=\{f_1(b,w')\phi(w)-f_1(b,w)\phi(w')\}\operatorname{cosec}\beta,\end{aligned}$$

by (4.3), if $\sin\beta\neq0$. If u were a double zero of $\phi(w)$, and $w=u+iv$, $w'=u-iv$, the right-hand side would be $O(v^2)$, which cannot be true of the left-hand side. Hence all the zeros of $\phi(w)$ are simple. If $\sin\beta=0$, the above expression is equal to

$$\pm\{f_2(b,w)\phi(w')-f_2(b,w')\phi(w)\},$$

and the result again follows.

5. Asymptotic formulae

Let p now denote $p(y)$, etc., and let

$$\lambda(x)=\tfrac{1}{2}\int_a^a(p+r)\,dy.$$

Let us write temporarily $w(x-y)=\theta$, $\lambda(x)-\lambda(y)=\phi$. Then

$$\begin{aligned}\int_a^x\cos(\theta-\phi)(qf_1+rf_2-wf_2)\,dy&=\int_a^x\cos(\theta-\phi)f_1'\,dy\\&=f_1(x)-\cos\{w(x-a)-\lambda(x)\}f_1(a)-\int_x^x\left(w-\frac{p+r}{2}\right)\sin(\theta-\phi)f_1\,dy,\end{aligned}$$

and

$$\int_a^x \sin(\theta-\phi)(pf_1+qf_2-wf_1)\,dy = -\int_a^x \sin(\theta-\phi)f_2'\,dy$$

$$= \sin\{w(x-a)-\lambda(x)\}f_2(a) - \int_a^x \left(w-\frac{p+r}{2}\right)\cos(\theta-\phi)f_2\,dy.$$

Hence

$$f_1(x) = \cos\{w(x-a)-\lambda(x)\}f_1(a) - \sin\{w(x-a)-\lambda(x)\}f_2(a) +$$

$$+\int_a^x \sin(\theta-\phi)\left(\frac{p-r}{2}f_1+qf_2\right)dy + \int_a^x \cos(\theta-\phi)\left(qf_1-\frac{p-r}{2}f_2\right)dy$$

$$= \sin\{w(x-a)-\lambda(x)+\alpha\} + \int_a^x (f_1\cos\theta+f_2\sin\theta)\chi\,dy +$$

$$+\int_a^x (f_1\sin\theta-f_2\cos\theta)\omega\,dy,$$

where

$$\chi = q\cos\phi-\tfrac{1}{2}(p-r)\sin\phi, \qquad \omega = q\sin\phi+\tfrac{1}{2}(p-r)\cos\phi.$$

Now

$$\int_a^x (f_1\cos\theta+f_2\sin\theta)\chi\,dy = \left[\frac{-\chi f_1\sin\theta}{w}\right]_a^x + \int_a^x \left(\frac{\chi' f_1+\chi f_1'}{w}+\chi f_2\right)\sin\theta\,dy$$

$$= \frac{1}{w}\chi(a)\sin\alpha\sin\{w(x-a)\} + \frac{1}{w}\int_a^x (\chi' f_1+\chi q f_1+\chi r f_2)\sin\theta\,dy,$$

and similarly

$$\int_a^x (f_1\sin\theta-f_2\cos\theta)\omega\,dy = \frac{1}{w}\omega(a)\cos\alpha\sin\{w(x-a)\} +$$

$$+\frac{1}{w}\int_a^x (-\omega' f_2+\omega p f_1+\omega q f_2)\sin\theta\,dy.$$

Hence

$$f_1(x) = \sin\{w(x-a)-\lambda(x)+\alpha\} + \frac{\chi(a)\sin\alpha+\omega(a)\cos\alpha}{w}\sin\{w(x-a)\} +$$

$$+\frac{1}{w}\int_a^x (\chi' f_1+\chi q f_1+\chi r f_2-\omega' f_2+\omega p f_1+\omega q f_2)\sin\theta\,dy. \quad (5.1)$$

There is a similar formula for $f_2(x)$, which may be obtained by changing the signs of q and w, interchanging p and $-r$, and changing the suffixes.

Let K denote the greatest of

$$|\chi(a)|+|\omega(a)|, \quad \max_{a\leqslant y\leqslant b}|\chi'+\chi q+\omega p|, \quad \max_{a\leqslant y\leqslant b}|-\omega'+\omega q+\chi r|$$

and of the corresponding functions in the equation for $f_2(x)$. Then, if $w=u+iv$,

$$|f_1(x)| \leqslant e^{|v|(x-a)}+\frac{K}{|w|}e^{|v|(x-a)}+\frac{K}{|w|}\int_a^x (|f_1|+|f_2|)e^{v(x-y)}\,dy.$$

Putting $f(x)=e^{|v|(x-a)}F(x)$, this gives

$$|F_1(x)| \leqslant 1+\frac{K}{|w|}+\frac{K}{|w|}\int_a^x (|F_1|+|F_2|)\,dy.$$

Let $$M_1=\max_{a\leqslant x\leqslant b}|F_1|, \qquad M_2=\max_{a\leqslant x\leqslant b}|F_2|.$$

Then $$M_1 \leqslant 1+\frac{K}{|w|}+\frac{K}{|w|}(b-a)(M_1+M_2),$$

and similarly for M_2. Hence

$$M_1+M_2 \leqslant 2+\frac{2K}{|w|}+\frac{2K}{|w|}(b-a)(M_1+M_2),$$

$$M_1+M_2 \leqslant \frac{2+2K/|w|}{1-2K(b-a)/|w|},$$

if $|w|>2K(b-a)$. Hence

$$f_1(x,w)=O(e^{|v|(x-a)}), \qquad f_2(x,w)=O(e^{|v|(x-a)}).$$

Substituting these in the right-hand side of (5.1), we obtain

$$f_1(x)=\sin\{w(x-a)-\lambda(x)+\alpha\}+O(e^{|v|(x-a)}/|w|), \tag{5.2}$$

$$f_2(x)=-\cos\{w(x-a)-\lambda(x)+\alpha\}+O(e^{|v|(x-a)}/|w|). \tag{5.3}$$

Similar results hold for $g_1(x)$ and $g_2(x)$ with a, α replaced by b, β.

Hence $$\phi(w)=\sin\{w(b-a)+\alpha-\beta\}+O(e^{|v|(b-a)}/|w|), \tag{5.4}$$

and so $$\frac{1}{\phi(w)}=\frac{1}{\sin\{w(b-a)+\alpha-\beta\}}+O\left(\frac{e^{-|v|(b-a)}}{|w|}\right) \tag{5.5}$$

in the region

$$\left|w-\frac{\beta-\alpha+n\pi}{b-a}\right| \geqslant \delta \quad (n=0,\pm 1,\ldots), \qquad |w| \geqslant \Delta \tag{5.6}$$

with any fixed small δ, and Δ large enough.

6. THEOREM. *Let the function ψ be integrable over (a,b), and of bounded variation in the neighbourhood of x, where $a<x<b$. Then the series* (3.9) *converges to the value* $\frac{1}{2}\{\psi(x+0)+\psi(x-0)\}$.

The leading terms in (5.2), etc., contribute to $\Phi_1(x,w)$

$$\frac{\sin\{w(x-b)-\lambda(x)+\beta\}}{\sin\{w(b-a)+\alpha-\beta\}}\times$$
$$\times\int_a^x[\psi_1\sin\{w(y-a)-\lambda(y)+\alpha\}-\psi_2\cos\{w(y-a)-\lambda(y)+\alpha\}]\,dy+$$
$$+\frac{\sin\{w(x-a)-\lambda(x)+\alpha\}}{\sin\{w(b-a)+\alpha-\beta\}}\times$$
$$\times\int_x^b[\psi_1\sin\{w(y-b)-\lambda(y)+\beta\}-\psi_2\cos\{w(y-b)-\lambda(y)+\beta\}]\,dy. \quad (6.1)$$

Now, if ψ is of bounded variation in $(x-\eta,x+\eta)$,

$$\int_a^x e^{-i\{w(y-a)-\lambda(y)+\alpha\}}\psi_1(y)\,dy=\frac{e^{-i\{w(x-a)-\lambda(x)+\alpha\}}}{-iw}\psi_1(x-0)+$$
$$+o(e^{|v|(x-a)}/|w|)+O(e^{|v|(x-\eta-a)}).$$

This is easily seen on applying the second mean-value theorem to the real and imaginary parts of

$$\int_{x-\eta}^x e^{-iwy}\{e^{i\lambda(y)}\psi_1(y)-e^{i\lambda(x)}\psi_1(x-0)\}\,dy.$$

Applying similar arguments to the other terms, we find that, in the upper half-plane, the first term in (6.1) is of the form

$$\frac{\sin\{w(x-b)-\lambda(x)+\beta\}}{\sin\{w(b-a)+\alpha-\beta\}}\left\{-\frac{e^{-i\{w(x-a)-\lambda(x)+\alpha\}}}{2w}\{\psi_1(x-0)+i\psi_2(x-0)\}+\right.$$
$$\left.+o(e^{v(x-a)}/|w|)+O(e^{v(x-\eta-a)})+O(e^{-v(x-a)})\right\}$$
$$=\frac{\psi_1(x-0)+i\psi_2(x-0)}{2w}+o\left(\frac{1}{w}\right)+O(e^{-\eta v})+O(e^{-2v(x-a)})$$

provided that w is in the region (5.6). Hence

$$\frac{1}{2\pi i}\int\Phi_1(x,w)\,dw,$$

taken round the upper half of the circle $|w|=N$, which lies entirely in the region (5.6), tends to the limit

$$\frac{1}{4}\{\psi_1(x+0)+\psi_1(x-0)\}+\frac{i}{4}\{\psi_2(x+0)+\psi_2(x-0)\}.$$

The integral round the lower half tends to the conjugate value. This, with the corresponding result for Φ_2, proves the theorem.

Results corresponding to other known theorems on Fourier series can also be obtained in a similar way.

The theory of these expansions can also be extended to include 'singular' cases in the same way as the ordinary Sturm-Liouville theory. The arguments required seem to be substantially the same in the two cases.

7. A particular case

If $r(x) = r$ is a constant, on eliminating f_2 from (1.2) we obtain

$$f_1''(x, w) = \{q^2+q'-(p-w)(r-w)\}f_1(x, w).$$

This gives

$$\frac{d}{dx}\{f_1'(x, w_m)f_1(x, w_n)-f_1(x, w_m)f_1'(x, w_n)\}$$
$$= (w_n-w_m)(w_m+w_n-p-r)f_1(x, w_m)f_1(x, w_n).$$

Hence

$$(w_n-w_m)\int_a^b (w_m+w_n-p-r)f_1(x, w_m)f_1(x, w_n)\,dx$$
$$= [f_1'(x, w_m)f_1(x, w_n)-f_1'(x, w_n)f_1(x, w_m)]_a^b$$
$$= (w_n-w_m)\left(\frac{\sin\beta\cos\beta}{k_n^2}-\sin\alpha\cos\alpha\right).$$

If $p(x) = -r$, $\beta = 0$, $0 < \alpha < \frac{1}{2}\pi$, $w_m \neq w_n$, this gives

$$\int_a^b f_1(x, w_m)f_1(x, w_n)\,dx = -\frac{\sin\alpha\cos\alpha}{w_m+w_n}.$$

Hence the functions $f_1(x, w_n)$ are not an orthogonal set in this case,

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